Fermilab

Effect of Space Charge on Instabilities

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Fermilab

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The inspiration for this presentation comes from

- ✓D. Möhl (Dispersion equation)
- ✓V. Kornilov, O. Boine-Frankenheim and I. Hofman (numeric simulations)
- ✓HB-2008 organizing committee

Dispersion Equation

Retain non-linearity only as shift of betatron frequencies with amplitudes =>

$$\frac{d^2 x_i}{dt^2} = -\Omega_i^{\,2}\left[Q_i^{\,2} x_i + 2Q_i \Delta Q_c \bar{x} + 2Q_i \Delta Q_{ic}\left(x_i - \bar{x}\right)\right] \qquad \text{where} \qquad \bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

=> $$x_i = -2\Omega_i^{\,2} Q_i \frac{\Delta Q_c - \Delta Q_{ic}}{\Omega_i^{\,2}\left(Q_i^{\,2} + 2Q_i \Delta Q_{ic}\right) - \omega^2}\bar{x}$$

ΔQ_c - coherent tune shift related to impedance, $F \propto -\Delta Q_c \bar{x}$

$\Delta Q_{ic} \equiv \Delta Q_{SC}$ - incoherent tune shift related to the space charge, $F \propto -\Delta Q_{ic}\left(x - \bar{x}\right)$

Integrating over particle distribution one obtains dispersion equation†

$$\boxed{\varepsilon(\omega) \equiv 1 - \Omega_0^{\,2} Q_0 \int \frac{\Delta Q_c(a_x, a_y, \omega) - \Delta Q_{ic}(a_x, a_y)}{\left(\Omega_b(a_x, a_y, \omega) + \Omega_0 \Delta Q_{ic}(a_x, a_y)\right)^2 - \left(\omega + i0 - n\Omega(p)\right)^2} \frac{df}{da_x} a_x^{\,2} a_y da_x da_y d\hat{p} = 0}$$

where $f = f(a_x, a_y, \hat{p})$, $\hat{p} = \frac{\Delta p}{p_0}$, $\int f\, a_x a_y da_x da_y d\hat{p} = 1$, $\eta = \alpha - 1/\gamma^2$,

$\Omega(p) = \Omega_0\left(1 - \eta\,\hat{p}\right)$ - dependence of revolution frequency on momentum

$\Omega_b(a_x, a_y, \omega) = \Omega_0\left[Q_0 + \left(\xi - Q_0\eta\right)\hat{p} + \Delta Q_{lat}(a_x, a_y)\right]$ - dependence of betatron frequency on particle betatron amplitudes and momentum (lattice only)

† D. Möhl, "On Landau damping of dipole modes by non-linear space charge and octupoles", CERN/PS 95-08, 1995

Dispersion Equation (2)

For $\Delta Q_c, \Delta Q_{ic} << Q_i$ expending to the first order in vicinity of unperturbed frequency $\omega_n = (n+Q_0)\Omega_0$ ($\omega = \omega_n + \delta\omega_n$) one finally obtains:

Impedance

Space charge

$$\boxed{\varepsilon_n(\delta\omega_n) \equiv 1 - \frac{1}{2}\int \frac{\Delta Q_c(a_x, a_y, \omega_n) - \Delta Q_{ic}(a_x, a_y)}{(\xi - (Q_0+n)\eta)\hat{p} + \Delta Q_{lat}(a_x, a_y) + \Delta Q_{ic}(a_x, a_y) - \frac{\delta\omega_n}{\Omega_0} - i0} \frac{df}{da_x} a_x{}^2 a_y da_x da_y d\hat{p} = 0}$$

Lattice

Additional notes

- $+i0$ - determines the rule to cross the pole for solution $\delta f(t,\theta) \propto \delta f \exp(-i(\omega t - n\theta))$
- Derivative over a_x appears due to tune change with horizontal amplitude[‡]. For the kernel which does not depend on a_x integration by parts results in

$$\varepsilon_n(\delta\omega_n) \equiv 1 + \int \frac{\Delta Q_c - \Delta Q_{ic}}{(\xi - (Q_0+n)\eta)\frac{\Delta p}{p_0} + \Delta Q_{lat} + \Delta Q_{ic} - \frac{\delta\omega_n}{\Omega_0} - i0} f\, a_x a_y da_x da_y d\left(\frac{\Delta p}{p_0}\right) = 0$$

- For zero lattice tune spread ($\xi - (Q_0+n)\eta = 0$, $\Delta Q_{lat}(a_x, a_y) = 0$) and $\Delta Q_c(a_x, a_y, \omega) = \Delta Q_c(\omega)$ one obtains $\delta\omega/\Omega_0 = \Delta Q_c((n+Q_0))$, i.e. the beam space charge is invisible

[‡] Hereward, "Landau damping by nonlinearity", CERN internal report MPS/DL 69-11, 1969

Linear Space Charge Model

- Assume:
  - KV-distribution => linear space charge field: $\Delta Q_{ic}(a_x, a_y) = Q_{ic}$
  - Zero lattice non-linearity: $\Delta Q_{lat}(a_x, a_y) = 0$
  - No wake dependence on amplitudes: $\Delta Q_c(a_x, a_y, \omega_n) = \Delta Q_{cn}$
- Denote
  - $\xi_n = \xi - (Q_0 + n)\eta$
- Integration over a_x and a_y results in the dispersion equation:

$$\varepsilon_n(\delta\omega_n) \equiv 1 + (\Delta Q_{cn} - \Delta Q_{ic}) \int \frac{f(\hat{p})d\hat{p}}{\xi_n \hat{p} + \Delta Q_{ic} - \frac{\delta\omega_n}{\Omega_0} - i0} = 0$$

- For Gaussian distribution that yields:

$$\varepsilon_n(\delta\omega_n) \equiv 1 + \frac{\Delta Q_{cn} - \Delta Q_{ic}}{\sqrt{2\pi}} \int \frac{\exp\left(-\hat{p}^2 / 2\sigma_p\right)}{\xi_n \hat{p} + \Delta Q_{ic} - \frac{\delta\omega_n}{\Omega_0} - i0} \frac{d\hat{p}}{\sigma_p} = 0$$

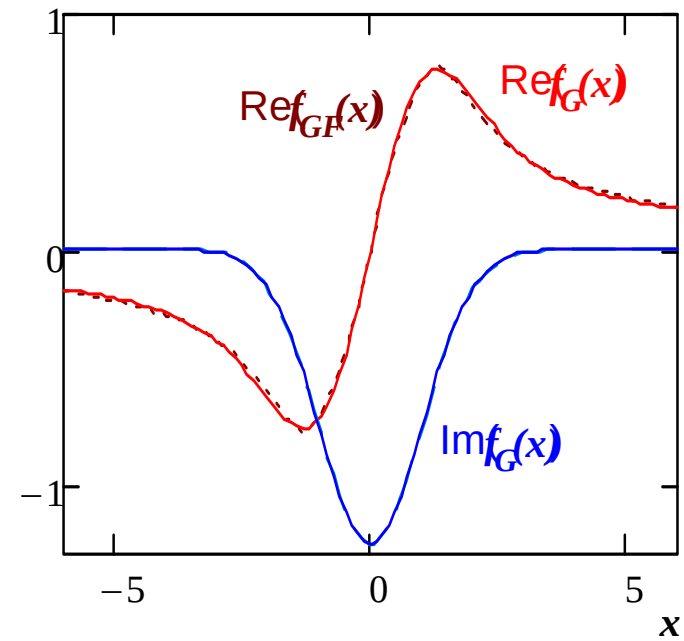
Linear Space Charge Model (2)

- Introduce the following function

$$f_G(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-y^2/2}}{x - y + i0} dy \equiv e^{-x^2/2} \int_0^x e^{y^2/2} dy - i\sqrt{\frac{\pi}{2}}\, e^{-x^2/2}$$

It can be approximated within ±5% as

$$f_G(x) \approx f_{GF}(x) \equiv \frac{x}{1+x^2} + 1.3\frac{x|x|^{1.6}}{5+|x|^5} - i\sqrt{\frac{\pi}{2}}\, e^{-x^2/2}$$



- One obtains for Gaussian distribution

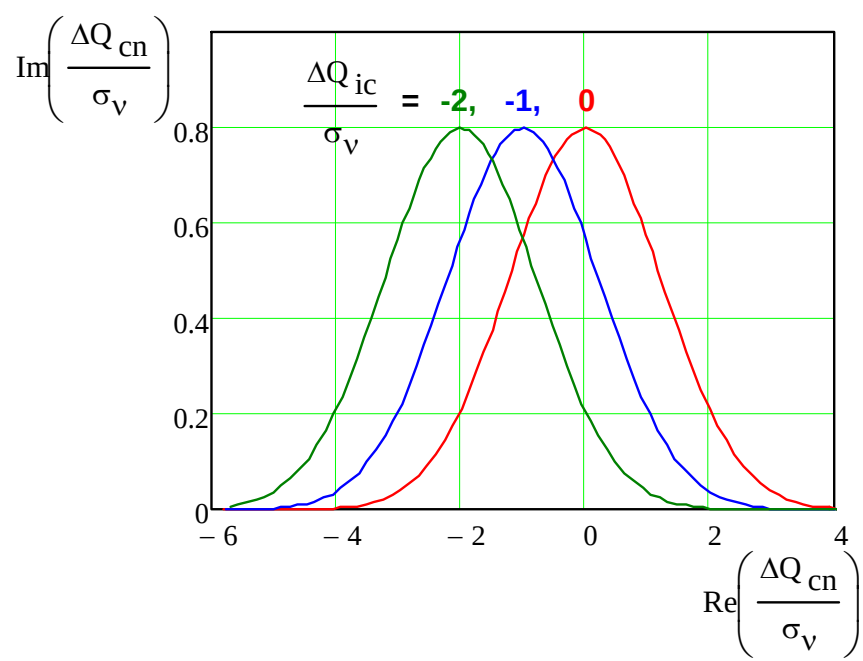
$$\varepsilon_n(\delta\omega_n) \equiv 1 - \frac{\Delta Q_{cn} - \Delta Q_{ic}}{\sigma_\nu} f_G\left(\frac{(\delta\omega_n/\Omega_0) - \Delta Q_{ic}}{\sigma_\nu}\right) = 0$$

where $\sigma_\nu = \sigma_p|\xi_n| \equiv \sigma_p|\xi_n + (n+1)\eta|$

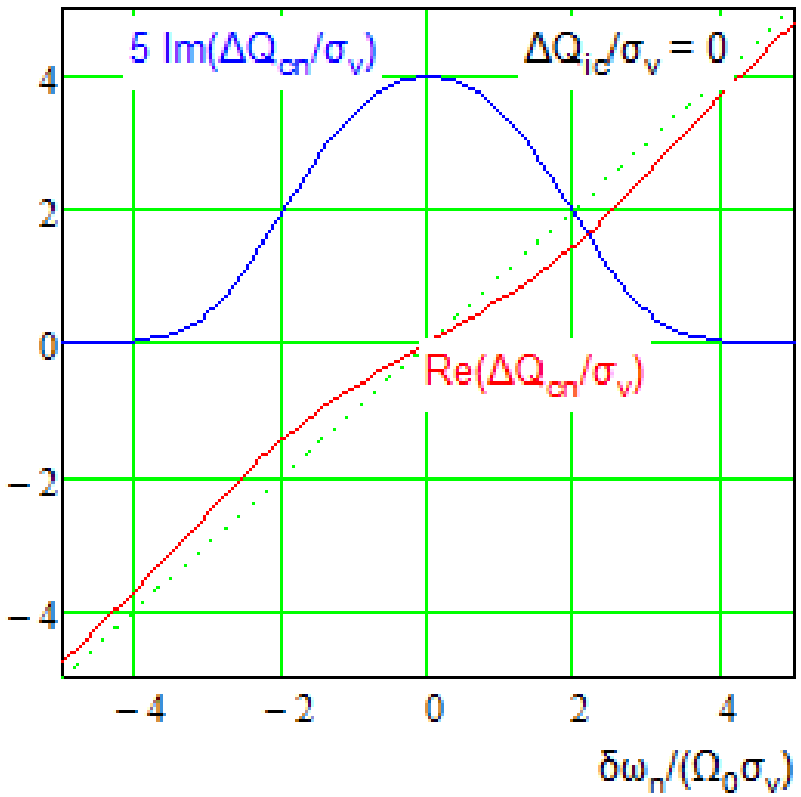
- The stability boundary is

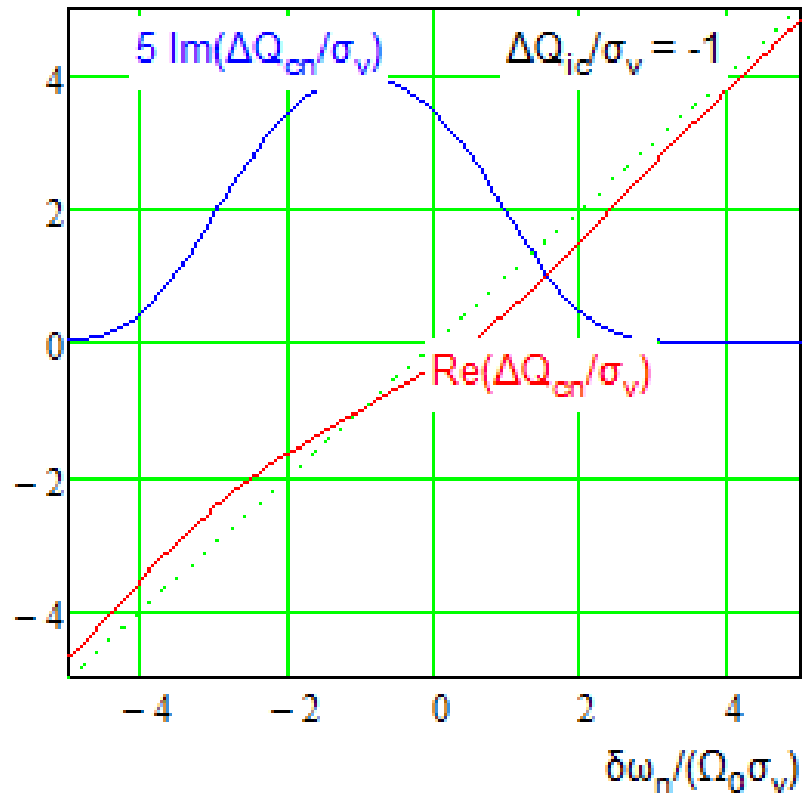
$$\frac{\Delta Q_{cn}}{\sigma_\nu} = \frac{\Delta Q_{ic}}{\sigma_\nu} + \frac{1}{f_G\left(\frac{(\delta\omega_n/\Omega_0) - \Delta Q_{ic}}{\sigma_\nu}\right)}$$

  - In the linear space charge model the space charge does not change the shape of stability boundary. It just shifts it by ΔQ_{ic}.



Linear Space Charge Model (3)





- Simple estimates resulting from the stability boundary calculations
  - Tune shift of coherent motion weakly affected by space charge and is close to the real part of coherent shift $\frac{\delta\omega_n}{\Omega_0} \approx \mathrm{Re}(\Delta Q_{cn})$
  - Landau damping: At maximum - $\lambda/\Omega_0 \approx \sqrt{2/\pi}\,\sigma_\nu$,

    In the tails - $\lambda/\Omega_0 \approx \sqrt{\pi/2}\sigma_\nu(\tilde{\nu}/\sigma_\nu)^2 \exp\left(-(\tilde{\nu}/\sigma_\nu)^2/2\right), \quad \tilde{\nu} = \delta\omega_n/\omega_0 - \Delta Q_{ic}$
    - Damping is not exactly proportional to particle density at coher. frequency

Space Charge Tune Spread for Gaussian beam

- Beam electric field is

$$E_x(x,y)=\frac{eN_p}{L_b}\int_0^\infty 4tx\frac{\exp\left(-\frac{x^2t^2}{1+2\sigma_x{}^2t^2}-\frac{y^2t^2}{1+2\sigma_y{}^2t^2}\right)}{\left(1+2\sigma_x{}^2t^2\right)^{3/2}\left(1+2\sigma_y{}^2t^2\right)^{1/2}}dt$$

- Perturbation theory results in for tune shift

$$\delta\nu_x=-\frac{1}{2\pi A_x{}^2}\left\langle X\delta P_x\right\rangle_{period},\quad X=\frac{x}{\sqrt{\beta_x}}=A_xe^{-i\mu_x},\quad P=-iA_xe^{-i\mu_x}$$

=>

$$\delta\nu_x(a_x,a_y)=-\frac{r_pN_p}{4\pi\beta^2\gamma^3}\frac{C}{L_b}\left\langle\beta_x\int_0^\infty\frac{4t\left[\mathrm{I}_0\left(\frac{\sigma_x{}^2a_x{}^2t^2}{2\left(1+2\sigma_x{}^2t^2\right)}\right)-\mathrm{I}_1\left(\frac{\sigma_x{}^2a_x{}^2t^2}{2\left(1+2\sigma_x{}^2t^2\right)}\right)\right]\mathrm{I}_0\left(\frac{\sigma_y{}^2a_y{}^2t^2}{2\left(1+2\sigma_y{}^2t^2\right)}\right)dt}{\left(1+2\sigma_x{}^2t^2\right)^{3/2}\left(1+2\sigma_y{}^2t^2\right)^{1/2}\exp\left(\frac{\sigma_x{}^2a_x{}^2t^2}{2\left(1+2\sigma_x{}^2t^2\right)}+\frac{\sigma_y{}^2a_y{}^2t^2}{2\left(1+2\sigma_y{}^2t^2\right)}\right)}\right\rangle_s$$

where a_x and a_y are dimensionless amplitudes of betatron motion:

$$x=\sigma_xa_x\cos\mu_x,\quad y=\sigma_ya_y\cos\mu_y.$$

Space Charge Tune Spread for Round Gaussian beam

- For round beam and smooth focusing

$$\beta = \beta_x = \beta_y\,, \quad \varepsilon = \varepsilon_x = \varepsilon_y\,, \quad \sigma_x = \sigma_y = \sqrt{\varepsilon\beta}$$

=>

$$\delta\nu_{SCx}(a_x,a_y) = -\delta\nu_{SC}\int_0^\infty 4t\,\frac{\left[\mathrm{I}_0\left(\frac{a_x^{\,2}t^2}{2(1+2t^2)}\right)-\mathrm{I}_1\left(\frac{a_x^{\,2}t^2}{2(1+2t^2)}\right)\right]\mathrm{I}_0\left(\frac{a_y^{\,2}t^2}{2(1+2t^2)}\right)}{(1+2t^2)^2\exp\left(\frac{a_x^{\,2}+a_y^{\,2}}{1+2t^2}\,\frac{t^2}{2}\right)}\,dt$$

where $\delta\nu_{SC} = \dfrac{r_p N_p}{4\pi\beta^2\gamma^3\varepsilon}\dfrac{C}{L_b}$

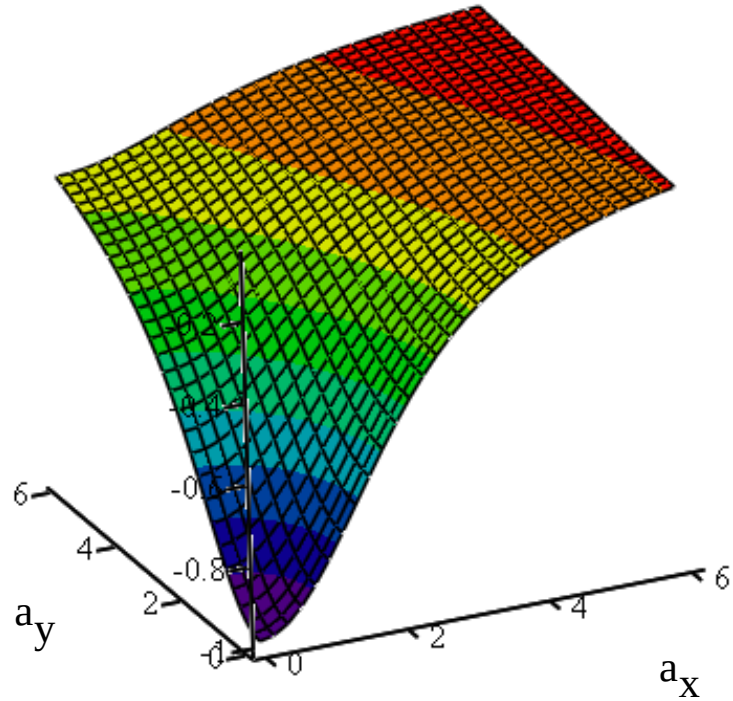
- The following expression fits tune shift within few percent

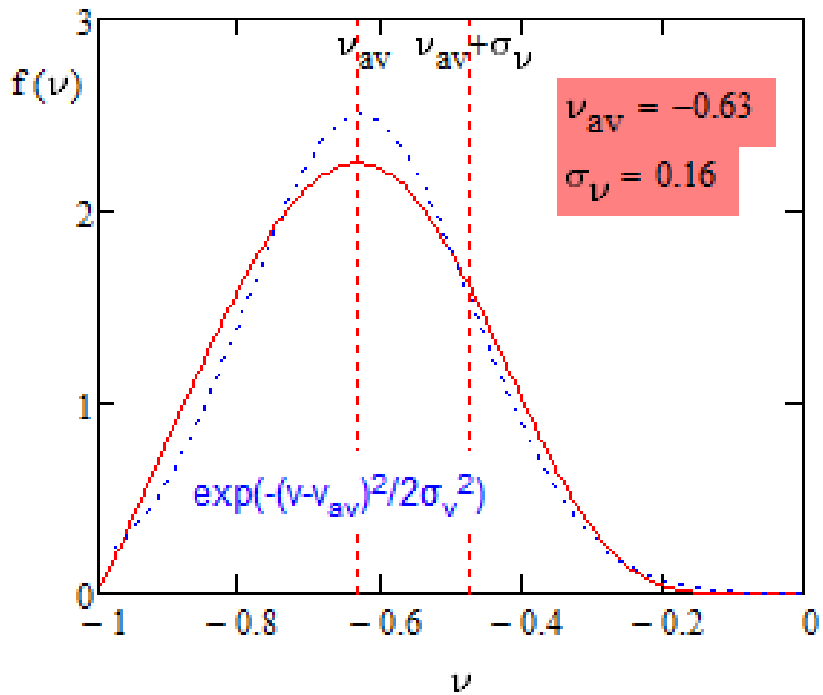
$$\delta\nu_{SCx}(a_x,a_y) \approx -\delta\nu_{SC}\,\frac{200+30a_y-10a_x-7a_xa_y}{200+40a_x^{\,2}-10a_x+30a_y^{\,2}+30a_y}\,,$$

$$a_x, a_x \le 6\,.$$

- Tune distribution function due to space charge tune shifts only

$$f(\nu) = \int \delta\big(\delta\nu(a_x,a_y)-\nu\big)f(a_x,a_y)a_xa_y\,da_x\,da_y$$





Effect of Space Charge on Landau Damping

- Assume:
  - Round Gaussian beam and smooth focusing, $\beta_x = \beta_y = const$.
  - Zero lattice non-linearity: $\Delta Q_{lat}(a_x, a_y) = 0$
  - No wake dependence on amplitudes: $\Delta Q_c(a_x, a_y, \omega_n) = \Delta Q_{cn}$
- Denote
  - $\xi_n = \xi - (1+n)\eta$
  - $\sigma_{\nu n} = |\xi_n| \sigma_p$
- For Gaussian distribution that yields the dispersion equation:

$$\varepsilon_n(\delta\omega_n) \equiv 1 + \frac{1}{2}\int \frac{\Delta Q_{cn} - \delta\nu_{SCx}(a_x, a_y)}{\xi_n \hat{p} + \delta\nu_{SCx}(a_x, a_y) - \frac{\delta\omega_n}{\Omega_0} - i0} \frac{a_x^{\,3} a_y}{\sqrt{2\pi}} \exp\left(-\frac{\hat{p}^2}{2\sigma_p^{\,2}} - \frac{a_x^{\,2} + a_y^{\,2}}{2}\right) da_x da_y \frac{d\hat{p}}{\sigma_p} = 0$$

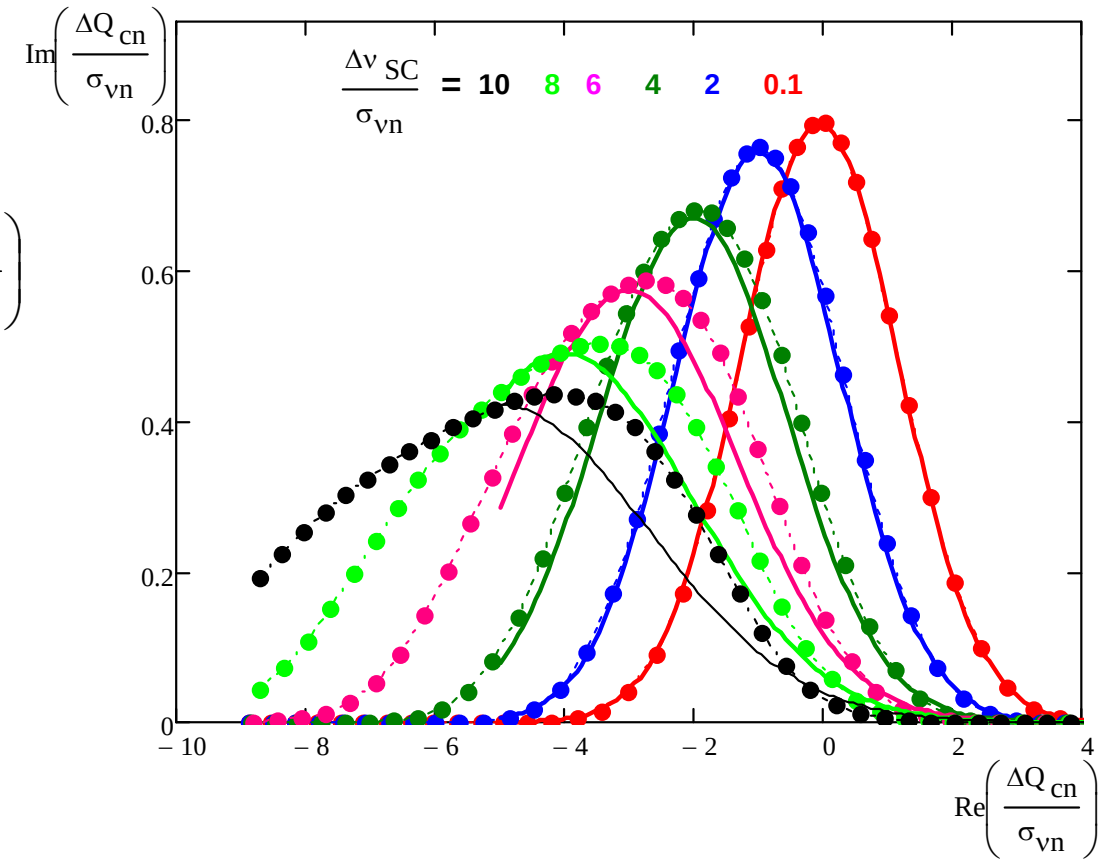
Effect of Space Charge on Landau Damping (2)

- **Stability boundary can be approximated by following expression (solid lines)**

$$\frac{\mathrm{Im}(\Delta Q_{cn})}{\sigma_{vn}} = \frac{0.97\sigma_{vn}}{\sigma_{tot}} \exp\left(-\frac{(\mathrm{Re}(\Delta Q_{cn}) - 0.5\,\delta\nu_{SC})^2}{2\sigma_{tot}^{\ 2}}\right)$$

where

$$\sigma_{tot} = 1.22\sqrt{\sigma_{vn}^{\ 2} + (0.16\delta\nu_{SC})^2}$$



- i.e. contrary to the linear space charge case
  - the stability diagram is shifted only by half of the linear tune shift ($\delta\nu_{SC}$)
  - The incoherent tune spread introduced by space charge widens the stability diagram in accordance with its rms value (0.16 $\delta\nu_{SC}$)

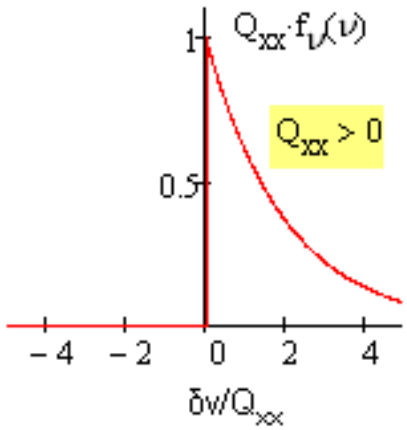
Improvement of Beam Stability by Cubic Nonlinearity

- Assume:
  - Round Gaussian beam and smooth focusing, $\beta_x = \beta_y = const$.
  - No wake dependence on the amplitudes: $\Delta Q_c(a_x, a_y, \omega_n) = \Delta Q_{cn}$
  - Contribution of tune chromaticity to the tune spread is much smaller than contribution from octupoles
  - Cubic lattice non-linearity
    - In the general case a combination of normal and skew-octupoles creates ,

$$\Delta Q_{lat} = Q_{xx} a_x^{\ 2} + Q_{xy} a_y a_y + Q_{yy} a_y^{\ 2}$$

    - but we consider the case when only $Q_{xx} \neq 0$

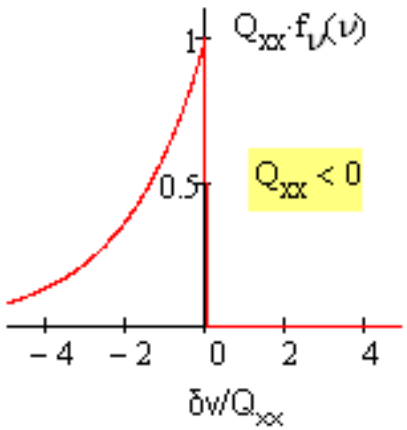
$$\Delta Q_{lat} = Q_{xx} a_x^{\ 2}$$



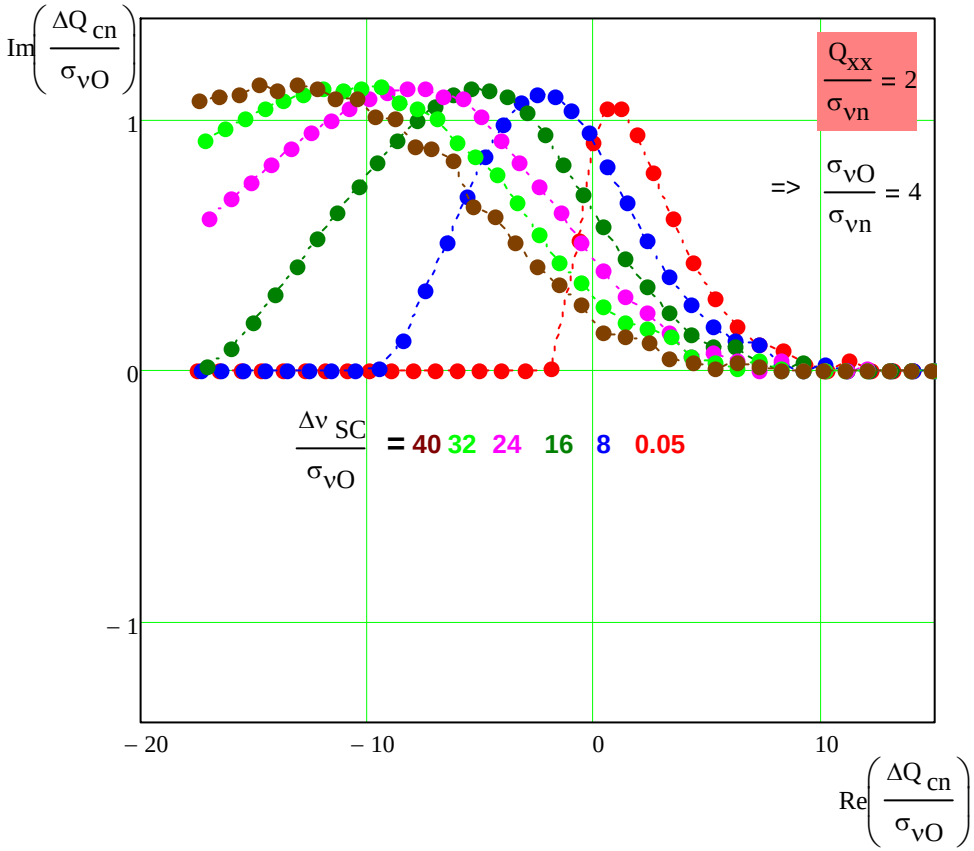
- Then the tune distribution is set by octupoles
  - For Gaussian beam it is

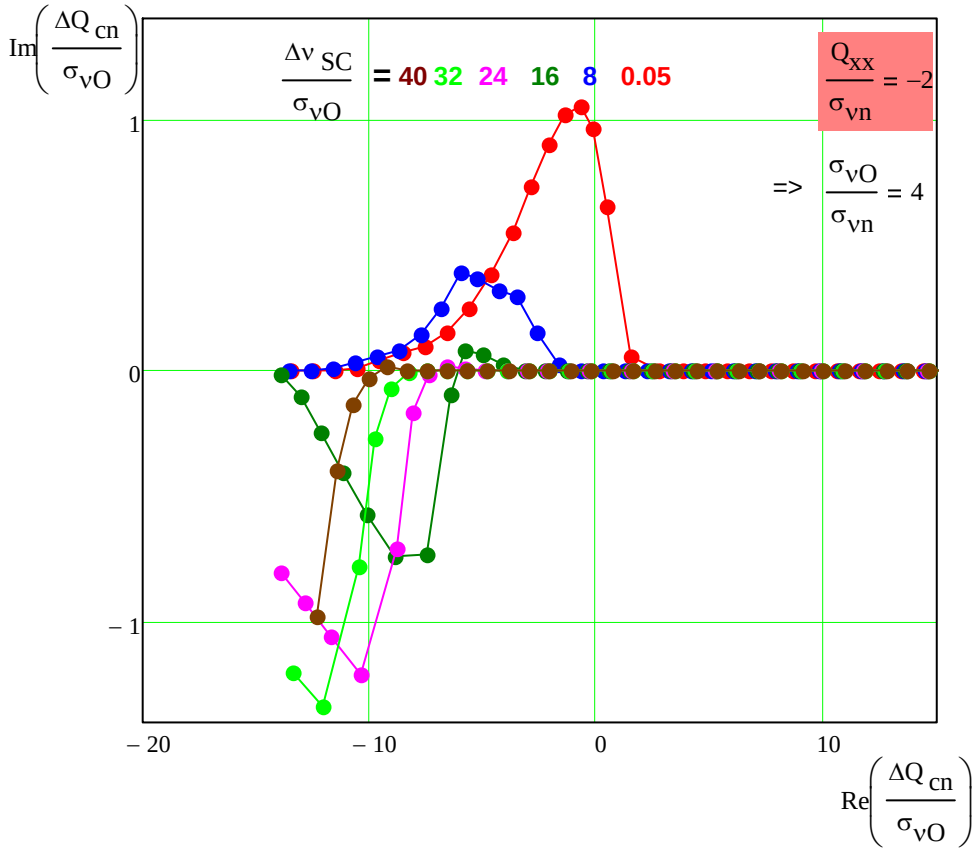
$$f_\nu(\nu) = \frac{\mathrm{sign}(\nu)}{2Q_{xx}} \exp\left(-\frac{\nu}{2Q_{xx}}\right), \quad Q_{xx} \geq 0$$

  - Rms tune spread $\boxed{\sigma_{\nu O} = 2Q_{xx}\ , \quad \bar{\nu} = 2Q_{xx}}$



Improvement of Beam Stability by Cubic Nonlinearity (2)





Stability boundary for positive and negative octupole non-linearity; chromatic rms tune spread is 4 times smaller than rms tune spread due to octupoles

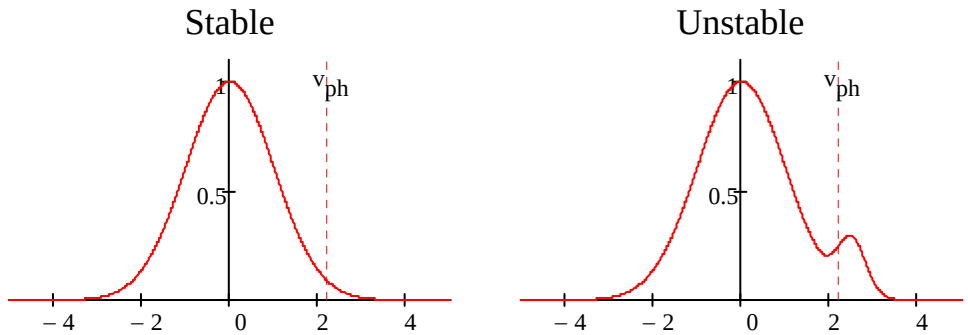
- Positive octupole nonlinearity improves beam stability
- Negative octupole nonlinearity can introduce Landau antidumping

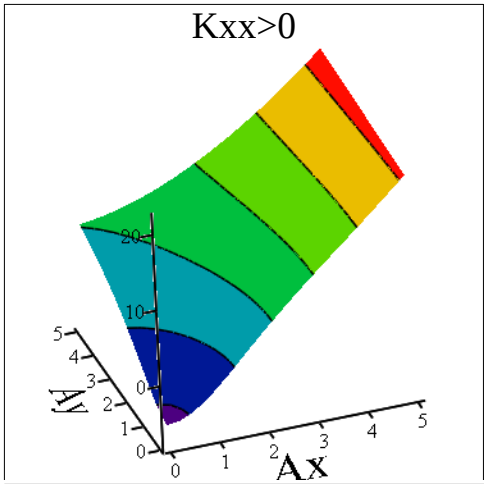
Improvement of Beam Stability by Cubic Nonlinearity (3)

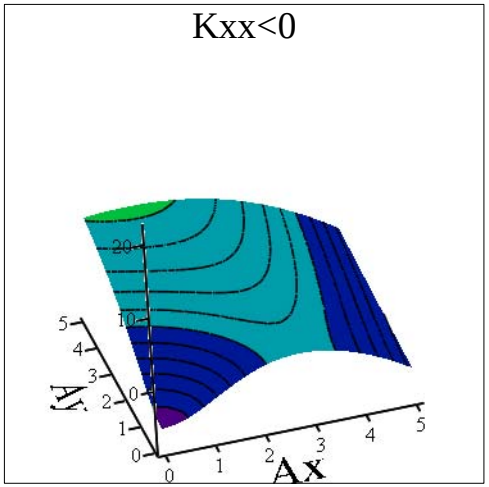
Landau Antidamping

- In plasma

$$\lambda_{Landau} \propto -\left.\frac{df}{dv}\right|_{v=v_{ph}}$$







- It is not quite similar or straightforward in the beam case

$$\varepsilon_n(\delta\omega_n) \equiv 1 - \frac{1}{2}\int \frac{\Delta Q_{cn} - \Delta Q_{ic}(a_x, a_y)}{K_{xx}a_x^{\,2} + \Delta Q_{ic}(a_x, a_y) - \dfrac{\delta\omega_n}{\Omega_0} - i0}\,\frac{df}{da_x}\,a_x^{\,2}a_y\,da_x\,da_y\,dp$$

- For above Eq. and $\mathrm{Im}(\Delta Q_{cn}) = 0$ the damping corresponds to $\mathrm{Im}(\varepsilon) < 0$ at the mode frequency. The sign of $\mathrm{Im}(\varepsilon)$ is controlled by the sign of term $\Delta Q_{cn} - \Delta Q_{ic}(a_x, a_y)$ at the mode frequency
- Naive explanation is that in the case of Landau antidumping the dependence of incoherent tune shift is not monotonic function of amplitudes

Improvement of Beam Stability by Cubic Nonlinearity (4)

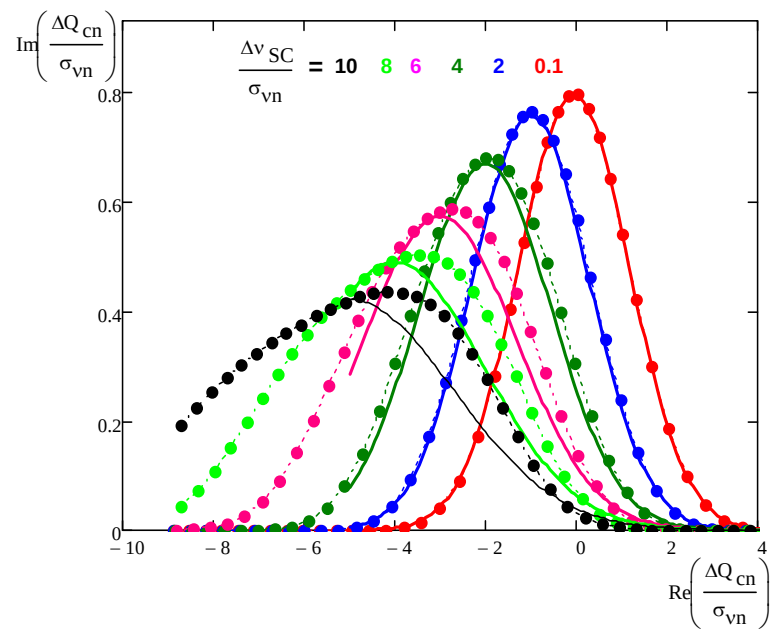
Comparison of stability regions created by chromatic tune spread

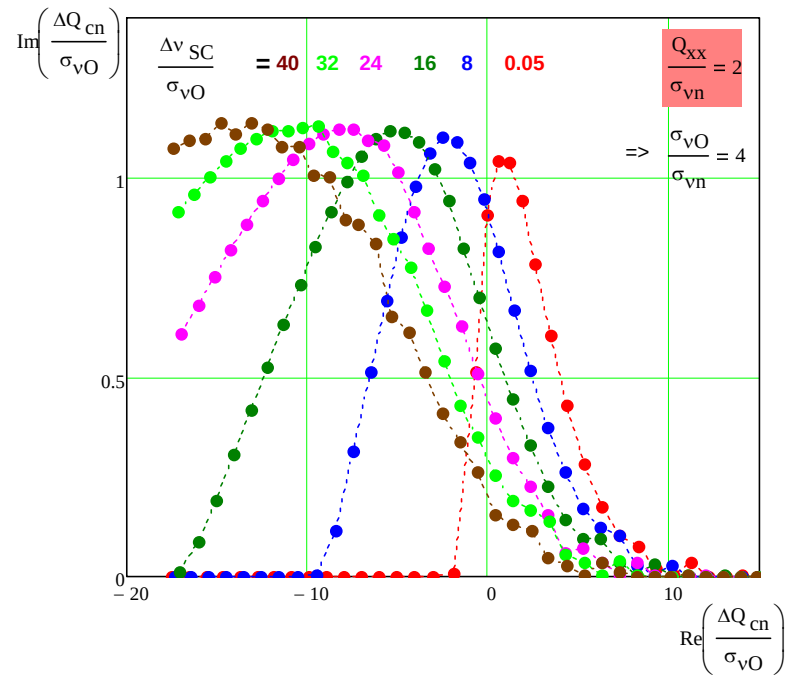
$$\sigma_{\nu n} = \sigma_p \xi_n \equiv \sigma_p \left(\xi + (n+1)\eta\right)$$

and tune spread due to octupoles

$$\sigma_{\nu O} = 2Q_{xx}$$

- Octupoles results in
  - 30% larger height and little dependence on the space charge
  - ~2 times larger width
  - Smaller shift of the stability region due to space charge (allows ~5 times larger $\Delta\nu_{SC}$)
  - No dependence on harmonic number
- Consequently, better beam stability for low harmonics
- But large space charge requires large octupole non-linearity which can reduce beam life time at large amplitudes





Impedances and Tune Shifts due to Vacuum Chamber

- We only consider the case of small beam, $\sigma_{beam} \ll a, b$
  ⇨ High order multipoles (> quad) excited by walls can be omitted
- By definition Incoherent tune shift ΔQ_{ic} is created by $F \propto -\Delta Q_{ic}\left(x-\bar{x}\right)$
  - It has two contributions
    - Direct space charge $\Delta Q_{ic}(a_x, a_y) \equiv \delta\nu_{SCx}(a_x, a_y)$
    - Incoherent tune shift due to quadrupole field excited by beam interaction with vacuum chamber walls
      - Wall incoherent tune shift - ΔQ_{icW}
- By definition Coherent tune shift ΔQ_c is created by $F \propto -\Delta Q_c \bar{x}$
  - It also has two contributions
    - From Dipole field – it is excited by transverse impedance

$$E_{\perp eff} = -iI_b \bar{x}\left(Z_\perp(\omega_n)/C\right) \Rightarrow \Delta Q_c = -i\frac{r_p N}{2\pi\beta\gamma\nu}\frac{Z_\perp(\omega_n)}{Z_0}$$

    - From quadrupole field excited by
      - Wall incoherent tune shift - ΔQ_{icW}

Transverse Impedance

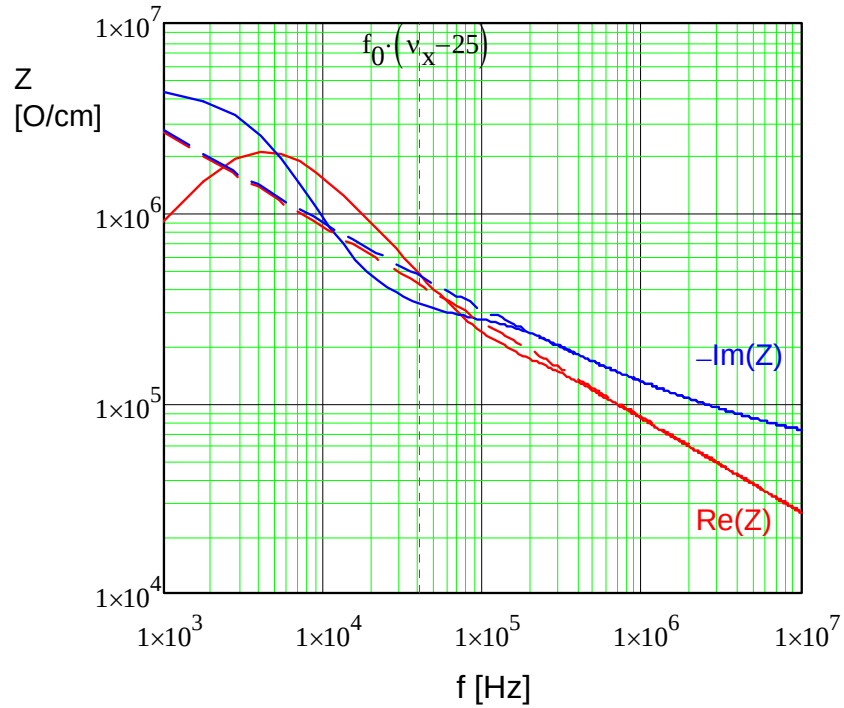
Impedance of Round Chamber

- At low frequencies, f≤100 MHz, the impedance is dominated by wall resistivity and interaction with vacuum chamber walls
- For sufficiently low frequency, $d\,\delta \le a^2$:

$$Z_{\perp R}(\omega) = \frac{Z_0 C}{2\pi a^2 \beta}\left(\frac{c\beta^2\left(\Theta(\omega)-i\right)}{a\sqrt{2\pi\sigma_R\omega}} - \frac{i}{\gamma^2}\right) \qquad (1)$$

where $\Theta(x)$ is the Heaviside step function

  - Effect of wall thickness can be important at small frequencies



$Z_\perp$ for FNAL Recycler if vacuum chamber would be round;
solid lines are for finite thickness stainless steel chamber built with exact formula: a=22 mm, d=2 mm, C=3.33 km, E_k=8 GeV
dashed lines are built with Eq. (1)
*Finite thickness of the wall becomes important at **f ≤ 100 kHz***

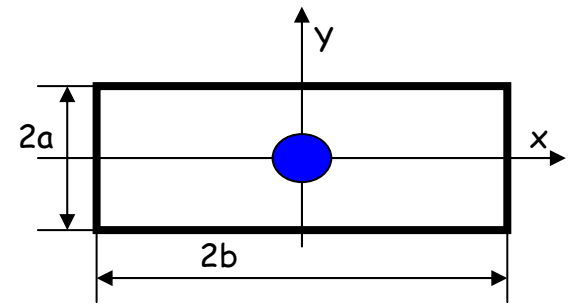
Transverse Impedance (2)

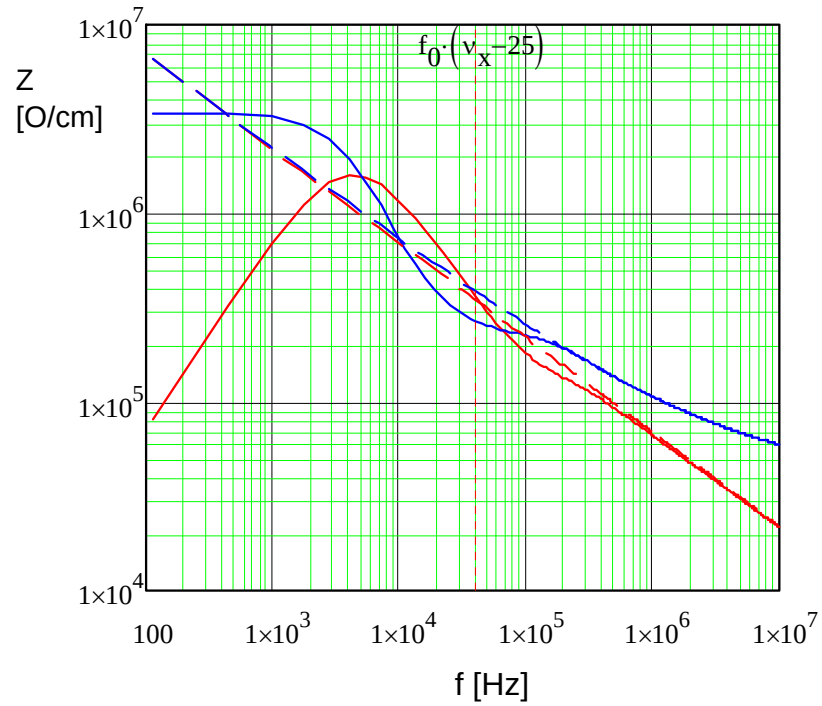
Impedances of flat vacuum chamber

- For a<<b one can approximate a vacuum chamber by two conducting planes at distance 2a
- Then the impedances are[§]

$$\begin{bmatrix} Z_y(\omega) \\ Z_x(\omega) \end{bmatrix} = \frac{\pi^2}{24}\begin{bmatrix} 2 \\ 1 \end{bmatrix} Z_{\perp R}(\omega) \approx \begin{bmatrix} 0.822 \\ 0.411 \end{bmatrix} Z_{\perp R}$$

  - Both impedances are proportional to the impedance of round chamber
    - Horizontal one is 2 times smaller than for the vertical one
- Similar to the round chamber the above equations are not applicable at low frequencies where $d\,\delta \geq a^2$





Z_y for FNAL Recycler; all parameters are the same as on the previous page solid curves are built with exact formulas for flat chamber

[§] Yokoya, 1993

Incoherent Tune Shifts due to Interaction with Walls

Tune shifts related to particle displacement relative to the beam center

Round Vacuum Chamber

- Beam displacement excites dipole field in a round vacuum chamber
- There are no incoherent tune shifts

Flat chamber case

- Summing multiple reflections of beam current in walls one obtains

$$E_y(y) = 2q\left(\sum_{n=1,3,5...}\left(\frac{1}{2an-\bar{y}-y}-\frac{1}{2an+\bar{y}+y}\right)-\sum_{n=2,4,6...}\left(\frac{1}{2an+\bar{y}-y}-\frac{1}{2an-\bar{y}+y}\right)\right)=\frac{\pi^2}{12}\frac{q}{a^2}\left(2\bar{y}+y\right)$$

- This can be rewritten as

$$E_y(y) = \frac{\pi^2}{12}\frac{q}{a^2}\left(\bar{y}+y\right)=\frac{\pi^2}{12}\frac{q}{a^2}\left((2+1)\bar{y}+\left(y-\bar{y}\right)\right)$$

Impedance + ΔQ_{icW} $\equiv \Delta Q_c$

ΔQ_{icW}

- That results the wall incoherent tune shift (Laslett tune shift)

$$\begin{bmatrix}\Delta Q_{icWX}\\ \Delta Q_{icWY}\end{bmatrix}=\frac{1}{96}\frac{Nr_pC}{a^2\beta^2\gamma}\begin{bmatrix}1/\nu_x\\ -1/\nu_y\end{bmatrix}$$

Incoherent Tune Shifts due to Interaction with Walls (2)

Incoherent Tune Shifts due to interaction with steel of dipoles

- Beam interaction with steel of dipoles creates DC magnetic field
  - It is not affected by beam betatron motion
- Therefore these tune shifts can be related to the lattice tune shifts
  - They can be easily compensated by intensity dependent tune correction

$$\begin{bmatrix} \Delta Q_{icD_X} \\ \Delta Q_{icD_Y} \end{bmatrix} = \frac{1}{48} \frac{N r_p C}{a_d^{\,2} \gamma} \eta_D \begin{bmatrix} 1/\nu_x \\ -1/\nu_y \end{bmatrix}$$

- Note that this tune shift can also be ascribed to the wall incoherent tune shift
  - As will be seen below it would not change the dispersion equation

Incoherent Tune Shifts due to Interaction with Walls (3)

Effect of Incoherent Tune Shifts on Dispersion Equation

- Formally "DC" and "AC" incoherent tune shifts go to different parts of dispersion equation

Impedance + ΔQ_{icW}

Direct space charge + ΔQ_{icW}

$$\varepsilon_n(\delta\omega_n) \equiv 1 - \frac{1}{2}\int \frac{\Delta Q_c(a_x, a_y, \omega_n) - \Delta Q_{ic}(a_x, a_y)}{(\xi - (Q_0 + n)\eta)\hat{p} + \Delta Q_{lat}(a_x, a_y) + \Delta Q_{ic}(a_x, a_y) - \frac{\delta\omega_n}{\Omega_0} - i0} \frac{df}{da_x} a_x^{\,2} a_y da_x da_y d\hat{p} = 0$$

Lattice + Qic from dipole

- In the final score it does not change the result, i.e. ΔQ_{icW} and ΔQ_d do not change the dispersion equation. They just shift tunes: $\delta\omega_n / \Omega_0 \to \delta\tilde{\nu}_n = (\delta\omega_n / \Omega_0) - \Delta Q_{icW} - Q_{icD}$

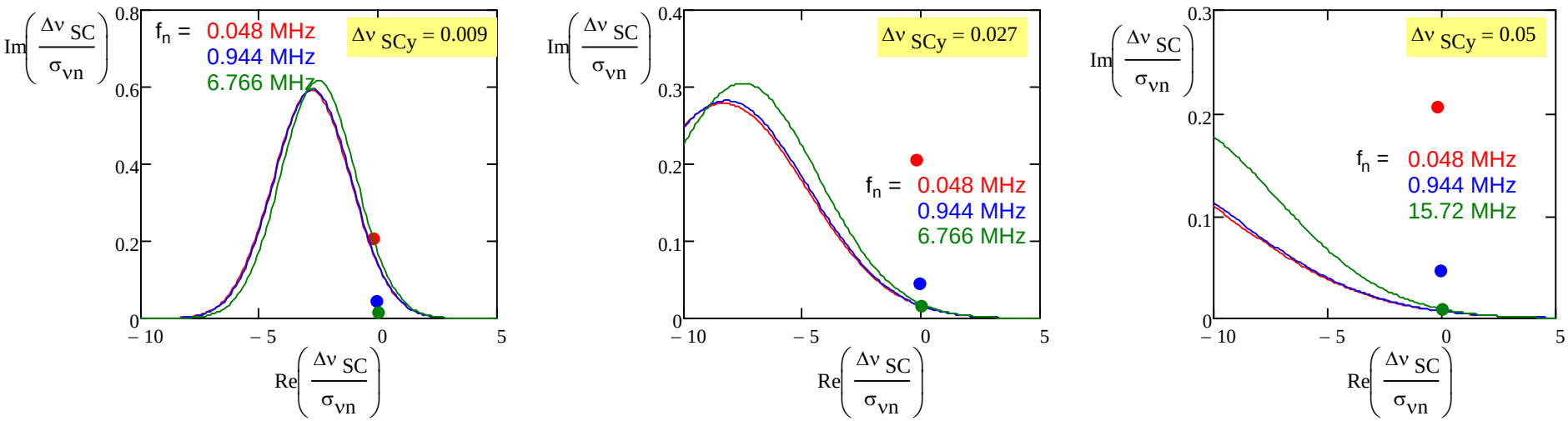
Impedance

Direct Space charge

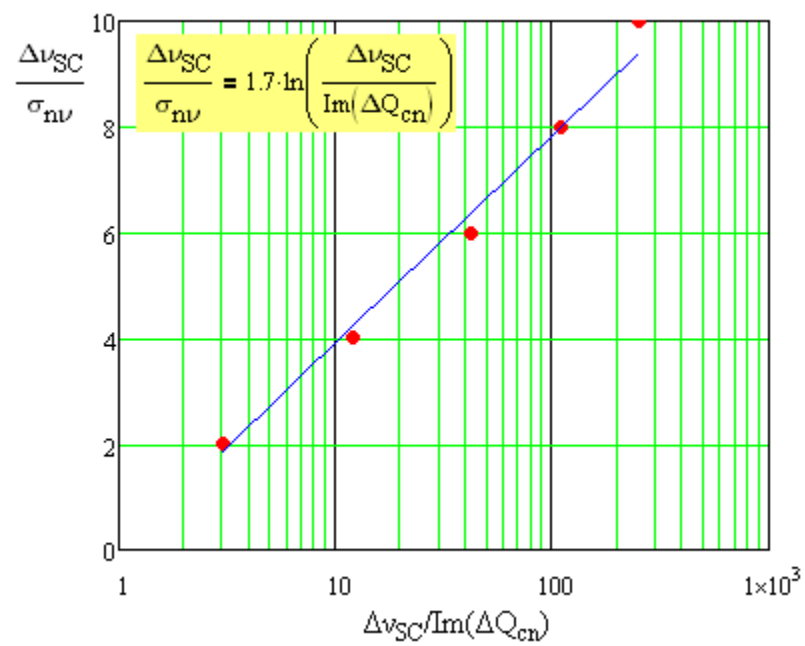
$$\varepsilon_n(\delta\tilde{\nu}_n) \equiv 1 - \frac{1}{2}\int \frac{\Delta Q_c(a_x, a_y, \omega_n) - \Delta Q_{ic}(a_x, a_y)}{(\xi - (Q_0 + n)\eta)\hat{p} + \Delta Q_{lat}(a_x, a_y) + \Delta Q_{ic}(a_x, a_y) - \delta\tilde{\nu}_n - i0} \frac{df}{da_x} a_x^{\,2} a_y da_x da_y d\hat{p} = 0$$

Lattice

Stability Boundary Estimates



- For γ >>1 $\mathrm{Re}(\Delta Q_{cn}) \sim \mathrm{Im}(\Delta Q_{cn})$.
  - Simple stability criterion can be found from stability diagram
  - Good coincidence with Recycler stability measurements



Practical recommendations

- For large space charge the stability boundary is usually determined by particle population in tails
  - Tails are not known in advance and do not stay constant in long run => An estimate can be the same accurate as detailed calculations
- If beam is bunched => picture is much more complicated
  - Chromatic tune spread => synchrotron freq. spread
  - Detuning wake, variation of betatron frequency along bunch
  - Single bunch and multi-bunch modes
    - If bunch length is about distance between bunches both signs of chromaticity stabilize instability
- Octupoles can improve beam stability but can adversely effect dynamic aperture
- Transverse damper suggests better way of instability stabilization
  - FNAL FPGA based dampers have bandwidth ~70 MHz
  - Higher frequencies are stabilized by chromatic tune spread
  - Booster damper is under commissioning now
    - Fast cycling synchrotron (15 Hz, 400 MeV $\rightarrow$ 8 GeV)

Conclusions

- Beam space charge separates coherent and incoherent tunes which strongly affects the beam stability
  - It is one of major limitations of the beam brightness
- Beam stabilization can be achieved by
  - Chromatic tune spread
  - Octupole non-linearity
  - Electronic feedback system
    - Frequencies above feedback band are suppressed by chromatic tune spread
    - Recent developments in electronics make it more promising
- Wall incoherent tune shift moves both coherent and incoherent tunes by the same amount resulting that it does not affect stability
- FNAL Recycler has very long bunch and therefore the developed formalism can be applied with little modifications
  - In general there is decent coincidence with estimates
  - Significant variability of stability boundary is observed
    - The reason is that the tails are responsible for beam stabilizations but their population depends on how the beam has been cooled