Theory for Neoclassical Toroidal Plasma Viscosity in a Toroidally Symmetric Torus

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Outline

- **Error fields and MHD activity distort the magnetic surface and break toroidal symmetry in $|B|$ in a toroidally symmetric torus.**
- **The asymmetric $|B|$ enhances neoclassical toroidal plasma viscosity.**
- **Enhanced toroidal plasma viscosity damps toroidal flow.**
- **Collisionality regimes derived from bounce averaged drift kinetic equation.**
- **Two classes of neoclassical plasma viscosity.**
- **Plateau regimes from drift kinetic equation.**
- **Conclusions.**

Magnetic Surface Distortion

- **A perturbed magnetic field B_1 modify $|B|$ directly [Smolyakov, PoP 1995]**

$$B = B_0 + B_1.$$

- **However, B_1 also perturbs the magnetic surface, and produces another modification on $|B|$ on the distorted magnetic surface [Shaing, PRL 2001, PoP 2003].**

- **On the perturbed magnetic surface,**

$$B = B_0(V, \theta, \zeta) + \xi^V (\partial B/\partial V) + \xi^\theta (\partial B/\partial \theta) + [\xi^\zeta (\partial B/\partial \zeta)],$$

where ξ^V, ξ^θ, and ξ^ζ are the contravariant components of plasma displacement vector ξ, (V, θ, ζ) are Hamada coordinates, θ is the poloidal angle, and ζ is the toroidal angle.

- **Thus, besides the direct change in $|B|$, the surface distortion produces another change in $|B|$ on the perturbed magnetic surface.**

- **The reason that |*B*| on the perturbed magnetic surface is relevant is because transport fluxes are calculated on the perturbed magnetic surface.**

- **A general expression for |*B*| on the perturbed magnetic surface is**

$$\boldsymbol{B} = B_0\,(1 - \varepsilon\cos\theta) -$$

$$B_0 \sum_{m,n} \left[b_{mnc}\cos(m\theta - n\zeta) + b_{mns}\sin(m\theta - n\zeta) \right]$$

where ε is the inverse aspect ratio, *m* is the poloidal mode number, and *n* is the toroidal mode number.

- **The perturbed |*B*| is no longer toroidally symmetric just like stellarators.**

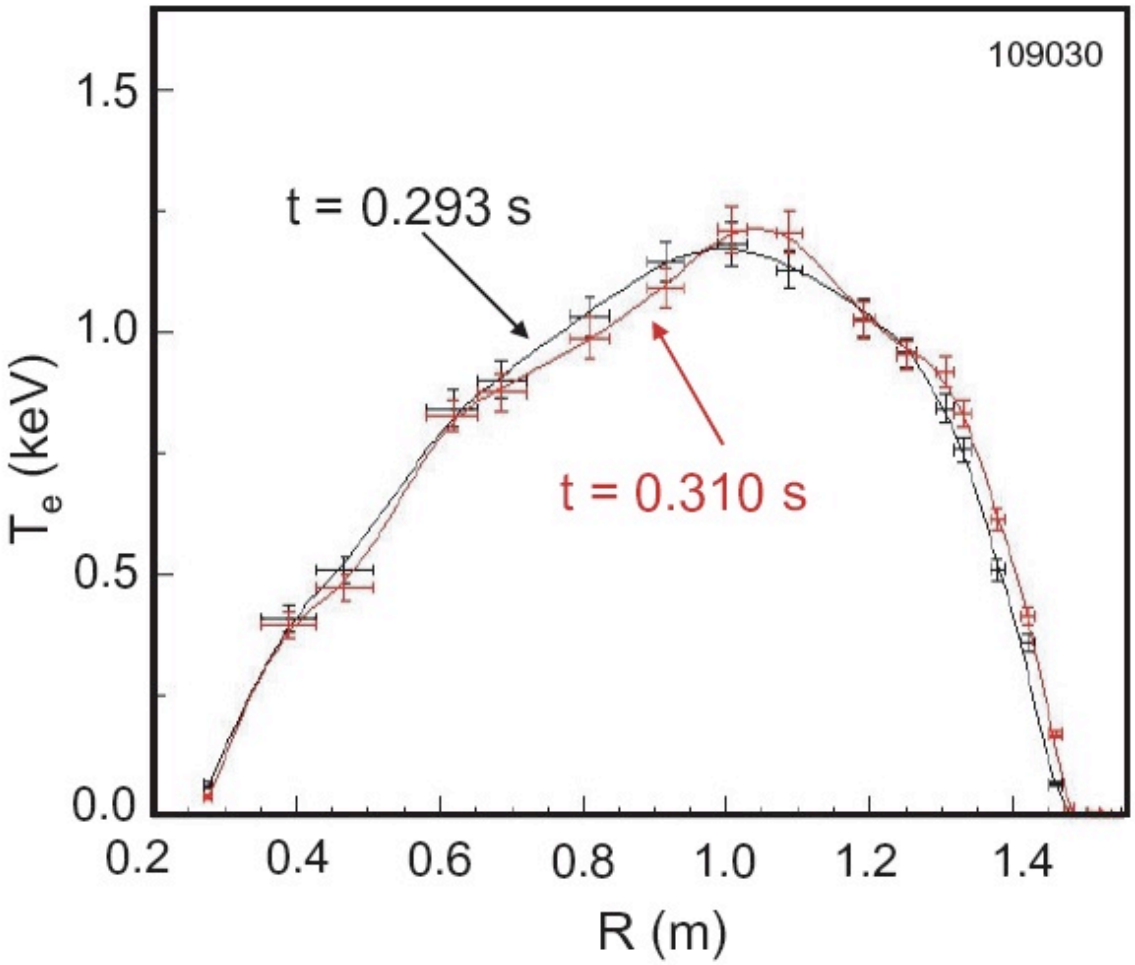
- **Stellarator physics becomes relevant to the confinement of a toroidally symmetric torus.**

- **However, the more relevant physics is not from the helically trapped particle orbits but from the wobbling banana orbits under the influence of the helically perturbed magnetic field.**

- **The reason is that magnitude of the perturbed magnetic field caused by error fields or MHD activities is not significant enough to trapped particles.**

- **The broken symmetry in $|B|$ results in enhanced neoclassical toroidal plasma viscosity (NTV).**

Distorted Electron Temperature Profile

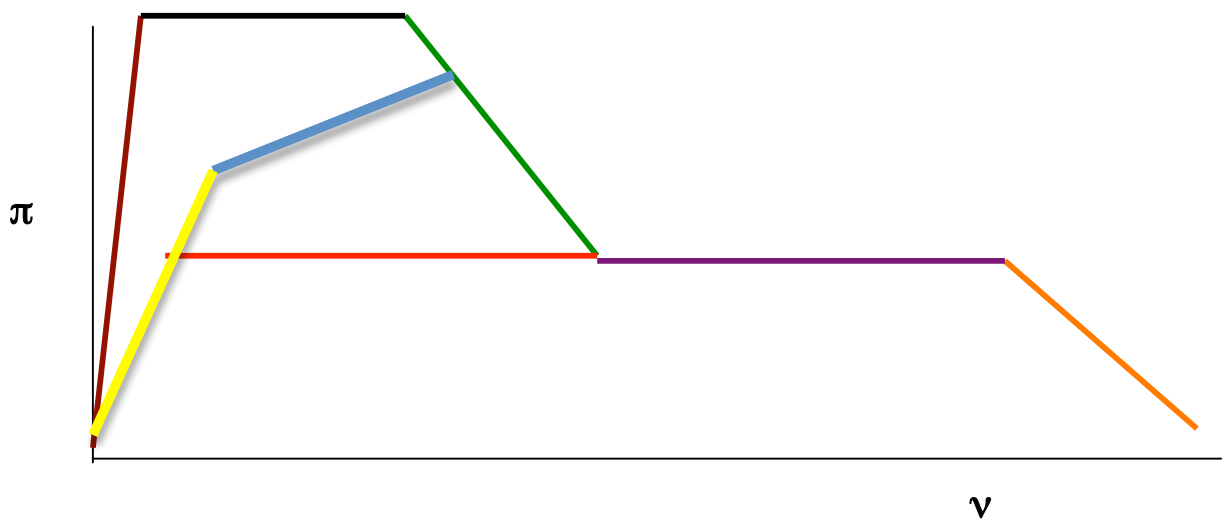


- **When MHD modes appear, electron temperature profile is distorted. This is an indication that the magnetic surface is distorted.**

*** S. E. Sabbagh, *et al.*, Nucl. Fusion 44, 560 (2004).**

Typical Collision Frequency Dependence of NTV

- **Typical collision frequency ν dependence of the toroidal viscous force π for ions (relative magnitudes are not accurate).**



Green line: 1/ν regime

Black line: Superbanana plateau regime

Brown line: Superbanana regime

Blue line: $\sqrt{\nu}$ regime (Colisional boundary layer)

Yellow line: ν regime (Collisionless detrapping/retrapping)

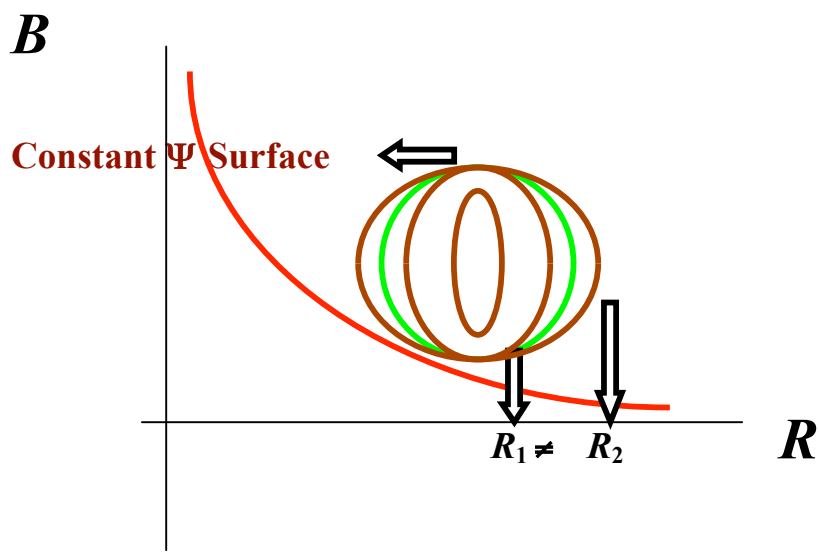
Red line: Plateau regime (Bounce-transit & drift resonance with mirror force for the equilibrium field only)

Purple line: Plateau regime (Transit and drift resonance with no mirror forces for all modes)

Orange line: Pfirsch-Schluter regime

Magnetic Surface Distortion in the Vicinity of an Island

- **When an island is imbedded in a tokamak, $|B|$ in the vicinity is distorted [Shaing, PRL 2001]:**



$$B/B_0 = 1 - \left[\frac{r_s}{R} \pm \frac{r_w}{R} \left(\overline{\Psi} + \cos\xi \right)^{1/2} \right] \cos\theta$$

$\overline{\Psi}$ **: Normalized helical flux function,**
ξ **:** $m\,(\theta - \zeta / q_S) + \omega t$**, helical angle,**
m **: Poloidal mode number,**
ζ **: Toroidal angle.**
r_w **: A measure of the width of the island. ($2\sqrt{2}\, r_w = w$)**

Enhanced Plasma Viscosity Caused by Asymmetric $|B|$

- **It is well known that the toroidal flow damping rate of the standard neoclassical toroidal plasma viscosity is too slow to explain what observed in tokamak experiments.**

- **The broken toroidal symmetry in $|B|$ enhances toroidal plasma viscosity and increases toroidal flow damping rate.**

- **For example, toroidal plasma viscosity (or asymmetric particle flux Γ) for the non-resonant error fields and MHD activity, in general, has the following form:**

$$\Gamma = -N \frac{\varepsilon^{1/2}}{4\sqrt{2}\pi^{3/2}} \left(\frac{Mc}{e\chi'}\right)^2 \mathrm{v}_t^4 \left[\lambda_1\left(\frac{p'}{p} + \frac{e\Phi'}{T}\right) + \lambda_2 \frac{T'}{T}\right],$$

where λs include physics of all collisionality regimes, N is plasma density, c is speed of light, M is the mass, v_t is the thermal speed, p is plasma pressure, e is electric charge, Φ is electrostatic potential, T is plasma temperature, χ is poloidal flux, and prime denotes radial derivative.

- **The asymmetric particle flux shown above is related to toroidal plasma viscosity through the flux-force relation for stellarators [Shaing, PF 1983]:**

$$\Gamma = -(c/e\chi'\psi')\langle B_p \bullet \nabla \bullet \pi\rangle = (c/e\chi'\psi')\langle B_t \bullet \nabla \bullet \pi\rangle,$$

where $B_p = -\chi'\nabla V \times \nabla\zeta$, $B_t = \psi'\nabla V \times \nabla\theta$ **.**

- **The thermodynamic force can also be cast in terms of toroidal flow** V^ζ **and poloidal flow** V^θ **:**

$$(p'/p + e\Phi'/T) = -(e/cT)(\chi' V^\zeta - \psi' V^\theta).$$

- **Notice that there is no explicit parallel mass flow in the thermodynamic forces. This is because the particle flux is calculated from the bounce averaged drift kinetic equation.**

- **Thus, toroidal plasma viscosity depends on both the toroidal flow and the poloidal flow.**

- **In general, toroidal flow evolution is coupled to that of the poloidal flow in all regimes.**

- **Neoclassical toroidal viscosity is observed in [Zhu, Sabbagh, Bell, et al., PRL 2006] and in JET [Sun, et al., EPS 2009].**

- **There is also a steady state toroidal flow associated with this viscosity. For example, in the 1/ν regime, it is:**

$$V_\zeta = \mathbf{3.54}\,\frac{cT_i}{e_i B_p}\,\frac{dT_i/dr}{T_i},$$

if the poloidal flow is the standard neoclassical value in the banana regime:

$$V_\theta = \mathbf{1.17}\,\frac{cT_i}{e_i B}\,\frac{dT_i/dr}{T_i}.$$

- **In the plateau regime, and the Pfirsch-Schluter regime, however, the steady state toroidal flow is quite small. [Shaing, PF 1986; PoP 2004]. In the large aspect ratio limit, the parallel flow is approximately zero.**

- **Next we discuss various regimes derived from bounce averaged drift kinetic equation.**

Bounce Averaged Drift Kinetic Equation

- **Bounce averaged drift kinetic equation is obtained by averaging drift kinetic equation over a banana orbit.**

- **It has the following form**

$$\left\langle \mathbf{v}_d \bullet \nabla \zeta_0 \right\rangle_b \frac{\partial f_{01}}{\partial \zeta_0} + \left\langle \mathbf{v}_d \bullet \nabla V \right\rangle_b \frac{\partial f_M}{\partial V} = \left\langle C(f_{01}) \right\rangle_b$$

where $\langle \bullet \rangle_b$ denotes bounce average, f_M is a Maxwellian distribution, $C(f)$ is the collision operator, $\left\langle \mathbf{v}_d \bullet \nabla V \right\rangle_b$ is the bounce averaged radial drift speed and $\left\langle \mathbf{v}_d \bullet \nabla \zeta_0 \right\rangle_b$ is the bounce averaged drift speed in the $\zeta_0 = q\theta - \zeta$ direction

$$\left\langle \mathbf{v}_d \bullet \nabla \zeta_0 \right\rangle_b = \frac{c\Phi'}{\chi'} - \frac{c\mu B_0}{e\chi'} \varepsilon' \left[\frac{2E(\kappa)}{K(\kappa)} - 1 \right],$$

- **Note that $\left\langle \mathbf{v}_d \bullet \nabla \zeta_0 \right\rangle_b$ consists of $E \times B$ drift and ∇B drift.**
- **$E(k)$ and $K(k)$ are complete elliptic integral of the 2nd and 1st kind respectively.**
- **k is the pitch angle parameter.**

Collisional 1/ν Regime

- **In this regime, the drift orbit trajectories of the trapped particles, i.e., wobbling bananas, are interrupted by collisions.**

- **The radial step size Δr is then**

$$\Delta r \sim \frac{V_{dr}}{\nu},$$

where V_{dr} is the bounce averaged radial drift velocity.

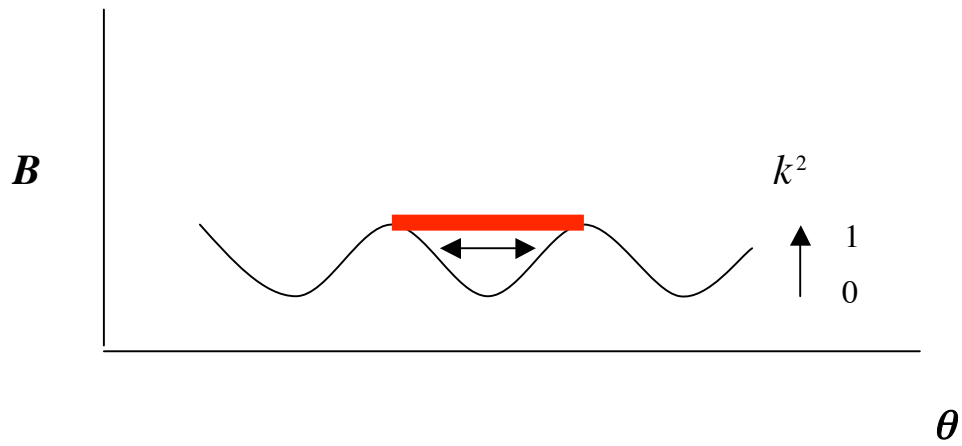
- **The transport flux thus scale as 1/ν based on random walk argument for a typical transport coefficient *D***

$$D \sim f \frac{(\Delta r)^2}{\tau},$$

because the decorrelation time τ is 1/ν and fraction of particles that participate in the transport processes *f* is $\sqrt{\varepsilon}$.

Boundary Layer

- **Schematic diagram of a trapped particle, i.e. a banana, in a toroidally symmetric torus.**



- **The pitch angle parameter k^2 varies between 0 (at the bottom of the well) and 1 (at the top of the well).**

- **In the vicinity of k^2 =1, there is a boundary layer.**

- **The boundary layer can be either collisional or collisionless.**

A. Collisional Boundary Layer

- **The width of the collisional boundary layer is determined by balancing the effective collision frequency in k^2 space and toroidal $E\times B$ drift frequency, i.e.,**

$$\frac{\nu/\varepsilon}{\left(\Delta k^2\right)^2} \sim q\,\omega_E$$

where q is the safety factor, ω_E is the poloidal $E\times B$ drift frequency and Δk^2 is the layer width in k^2 space.

- **The width of the collisional boundary layer in k^2 space is, thus, proportional to $\sqrt{\nu}$.**

- **This leads to a transport flux that scales as $\sqrt{\nu}$ because the fraction of particles that participate in the transport process is $\sqrt{\varepsilon}\,\Delta k^2$, the step size is V_{dr}/ω_E, and the de-correlation frequency is $(\nu/\varepsilon)/(\Delta\kappa^2)^2$.**

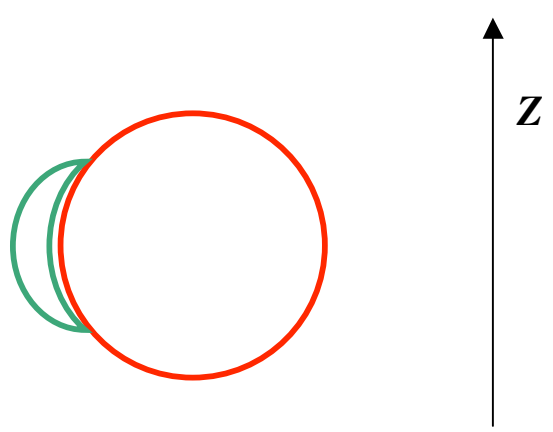
B. Collisionless Boundary Layer

- **The width of the collisionless boundary layer in k^2 space is determined by the ratio of the normalized magnitude of the perturbed field $\delta B/B$ to the inverse aspect ratio because the ratio is a measure of the asymmetry in $|B|$.**

- **Collisionless boundary layer width Δk^2 is $\sim(\delta B/B)/\varepsilon$.**

- **Particles inside the collisionless boundary layer can undergo collisionless detrapping and retrapping.**

- **Collisionless detrapping/retrapping is a process in which a trapped particle (green crescent shaped orbit) can become detrapped, i.e., circulating (red circle) without suffering collisions and vice versa.**



Collisionless detrapping/retrapping

- **It is a common phenomenon in a toroidal system without any symmetry because asymmetry makes the change of orbit topology possible for the lack of 2nd adiabatic invariant.**

- **The transport flux scales as ν when the collisionless boundary layer width dominates.**

- **The random argument for this scaling is the same as that for the collisional boundary layer except the layer width is replaced by $(\delta B/B)/\varepsilon$.**

Resonance between $E\times B$ Drift and ∇B Drift

- **When bounce averaged ∇B drift speed $\omega_{\nabla B}$ is comparable to $E\times B$ drift speed ω_E so that $\omega_E+\omega_{\nabla B}\approx 0$. There can be a resonance so that the radial step size is unbounded. This is unphysical and must be resolved.**

- **There are two ways to resolve this issue.**

- **One is to use collisions to remove the singularity.**

- **This leads to the superbanana plateau regime.**

- **The transport flux in this regime is dominated by particles that are centered around the resonance condition: $\omega_E+\omega_{\nabla B}\approx 0$ with a width Δk^2 determined by**

$$\omega_{\nabla B}(\Delta k^2) \sim \frac{\nu/\varepsilon}{\left(\Delta k^2\right)^2}.$$

- **Thus, $(\Delta k^2) \sim \left(\frac{\nu/\varepsilon}{\omega_{\nabla B}}\right)^{1/3}$.**

- **The radial step size is**

$$\Delta r \sim \frac{V_{dr}}{\omega_{\nabla B}\left(\Delta k^2\right)}.$$

- **The fraction of particles that participate in the transport processes is $\sqrt{\varepsilon}\Delta k^2$.**

- **The transport coefficient is thus independent of the collision frequency because**

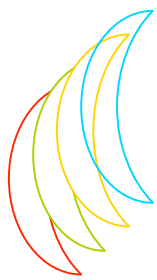
$$\boldsymbol{D} \sim \sqrt{\varepsilon}\ \frac{V_{dr}^2}{\omega_{\nabla B}},$$

- **This is analogous to the standard plateau regime in the conventional transport theory.**

- **The other method to remove the singularity is to use nonlinear orbit trajectory when the collision frequency is small.**

- **This leads to the superbanana regime.**

- **A superbanana consist of wobbling bananas that have zero toroidal drift speed at some angle to form a giant banana.**



Wobbling Bananas and Superbanana

- **The width of the superbanana is** $(\Delta r)_{sb} \sim r\sqrt{(\delta B/B)/\varepsilon}$.

- $(\Delta r)_{sb}$ **depends only on the geometry and is independent of the particle energy or the magnetic field strength.**

- **The fraction of the superbananas** f_{sb} **is of the order of** $f_{sb} \sim \sqrt{(\delta B/B)}$.

- **The diffusion coefficient or viscous coefficient scales like**

$$\boldsymbol{D} \sim \nu \frac{\sqrt{(\delta B/B)}}{\varepsilon} r^2.$$

- **It is a very large diffusion coefficient even with** $(\delta B/B) \sim 10^{-4}$**.**

- **It is the most dangerous transport flux in any toroidal systems.**

Two Classes of Neoclassical Toroidal Plasma Viscosity

- **Neoclassical toroidal plasma viscosity can be classified according to the collision operator employed in the calculations.**

- **There are two classes. One class involves the bounce averaged collision operator (i.e., momentum restoring term is not important), and the other does not (i.e., momentum restoring term is important).**

- **The $1/\nu$, the superbanana plateau, the superbanana, and the $\sqrt{\nu}$ regime, for example, belong to the first class. In this class, the momentum restoring term is not important. The thermodynamic forces that involves plasma flow can be cast in the following form:**

$$(p'/p + e\Phi'/T) = -(e/cT)(\chi' V^{\zeta} - \psi' V^{\theta})$$

Note that there is no explicit parallel flow dependence.

- **The nonlinear plasma viscosity (e.g., the one used to model L-H mode transition [Shaing, PRL 1989]), the bounce-drift resonance [Park, et al., PRL 2009], and the transit-drift resonance [Shaing, et al., PPCF, 2009] belong to the second class where the momentum restoring term becomes important. The thermodynamic forces, in this case, depend on the parallel mass flow.**

Drift Kinetic Equation with Flows

- **The linear drift kinetic equation that has the plasma flows as its driving term is [Shaing and Spong, 1991]**

$$\left[\left(\mathrm{v}_{\|}+V_{\|}\right)\boldsymbol{n}+\boldsymbol{V}_E\right]\bullet\nabla g-C(g)=$$

$$2\frac{\mathrm{v}^2}{\mathrm{v}_\mathrm{t}^2}\left(\frac{1}{2}-\frac{3}{2}\frac{\mathrm{v}_{\|}^2}{\mathrm{v}^2}\right)f_M\left(\frac{\boldsymbol{V}\bullet\nabla B}{B}-\frac{2}{5}L_1^{(3/2)}\frac{\boldsymbol{q}\bullet\nabla B}{Bp}\right),$$

- **Where *g* is the perturbed particle distribution,**

  $\mathrm{v}_{\|}$ **is the particle speed that is parallel to the magnetic field *B*,**

  $V_{\|}$ **is the plasma flow that is parallel to *B*,**

  ***n* is the unit vector in the direction of *B*,**

  $\boldsymbol{V}_E$ **is the *E*×*B* drift velocity,**

  $C(g)$ **is the test particle part of the collision operator,**

  v is the particle speed,

  $\mathrm{v}_\mathrm{t}=\sqrt{2T/M}$ **is the thermal speed,**

V **is the plasma flow velocity,**

q **is the plasma heat flow,**

p* = *NT **is the plasma pressure, and**

$L_1^{(3/2)}(x) = 5/2 - x$ **with** $x = \mathrm{v}^2/\mathrm{v}_\mathrm{t}^2$ **is the Laguerre polynomials.**

- **The momentum restoring term in the collision operator is taken into account in the parallel flow. This is the key difference between the drift kinetic equation presented and the conventional one.**

- **When we perform bounce average in the low collisionality regime, e.g., the 1/ν regime, the parallel plasma flow term is averaged to zero.**

- **Thus, the difference in the thermodynamic forces in the two classes of the neoclassical toroidal plasma viscosity is apparent.**

Bounce-Transit and Drift Resonance

- **The key assumption to obtain bounce-transit and drift resonance is that the mirror force term resulting from the equilibrium magnetic field is kept. But the mirror force terms caused by the perturbed magnetic field are not.**

- **No bounce average is required. Otherwise, the bounce motion will be bounce averaged out.**

- **Both trapped particles, i.e., bananas [Park, et al., PRL 2009], and circulating particles [Shaing, et al., PPCF 2009] contribute through the resonances between bounce frequency and the $E\times B$ precession drift frequency for the trapped particles and between transit frequency and the precession frequency for the circulating particles.**

- **The toroidal component of the viscous force resulting from the trapped particles is (neglecting heat flow for simplicity)**

$$\langle \boldsymbol{B}_t \bullet \nabla \bullet \pi \rangle = \mu_{p1} \boldsymbol{V} \bullet \nabla \theta + \mu_{t1} \boldsymbol{V} \bullet \nabla \zeta .$$

- **The viscous coefficients are**

$$\begin{pmatrix}\mu_{p1}\\ \mu_{t1}\end{pmatrix} = \boldsymbol{NM}\,\mathrm{v_t}\,\frac{\sqrt{\pi}}{2}\boldsymbol{B}\,\frac{\psi'}{\chi'}\sum_{n,l}\frac{1}{|l|}\int_{x_{\min}}^{\infty} dx x^2 e^{-x}\,\frac{K(k_0)}{\pi/2}\,\frac{1}{|dG_{nl}/dk^2|_{k_0}}\times$$

$$\begin{pmatrix}-\left(a_{nl}b_{-n-l}+q\,|b_{nl}|^2\right)\\ |b_{nl}|^2\end{pmatrix}_{k_0},$$

where a_{nl} and b_{nl} are Fourier coefficients of perturbed fields in terms of Jacobian elliptic function.

- **The contribution from the circulating particles has the same form except the viscous coefficients are different:**

$$\begin{pmatrix}\mu_{p1}\\ \mu_{t1}\end{pmatrix} = \boldsymbol{NM}\,\mathrm{v_t}\,\sqrt{\pi}\,\boldsymbol{B}\,\frac{\psi'}{\chi'}\sum_{n,l}\frac{1}{|l|}\int_{x_{\min}}^{\infty} dx x^2 e^{-x}\,\frac{K(k_0)}{\pi/2}\,\frac{k_0^2 F(k_0)}{|dH_{nl}/d(1/k^2)|_{k_0}}\times$$

$$\begin{pmatrix}-\left(c_{nl}d_{-n-l}+q\,|d_{nl}|^2\right)\\ |d_{nl}|^2\end{pmatrix}_{k_0},$$

where c_{nl} and d_{nl} are Fourier coefficients of perturbed fields in terms of Jacobian elliptic function.

- **This viscosity and the plasma viscosity in the plateau regime, where all mirror forces are neglected, have the same scaling, because both viscosities deal with the same kinetic terms in the drift kinetic equation. Only the treatment of the parallel particle speed is different.**

- **This viscosity is a nonlinear function of the radial electric field.**

- **The onset of the nonlinearity for the trapped particles is different from that for the circulating particles.**

- **For trapped particles, nonlinearity onsets when**

$$\sqrt{x_{\min}} = \left| \frac{cE_r}{B_p v_t} q \left| \frac{n}{l} \right| \sqrt{\frac{2}{\varepsilon}} \right| > 1.$$

- **For circulating particles, when** $\sqrt{x_{\min}} = \left| \frac{cE_r}{B_p v_t} q \left| \frac{n}{l} \right| \right| > 1$**,**

  nonlinearity onsets.

- **For the same mode numbers, contribution from the trapped particles becomes nonlinear at a smaller value of the radial electric field by a factor of** $\sqrt{2/\varepsilon}$**.**

- **The viscous force results from bounce-transit and drift resonance connects naturally to that in the Pfirsch-Schluter regime when collision frequency increases because they belong to the same class.**

Neoclassical Toroidal Plasma Viscosity Depends on Radial Electric Field Nonlinearly

- **It has been demonstrated that the bounce-transit and drift resonance induced NTV depends on the radial electric field nonlinearly.**

- **NTV in other regimes is a nonlinear function of the radial electric field as well.**

- **They either decrease exponentially or algebraically when the radial electric field becomes large.**

- **For example, the particle flux Γ in the superbanana plateau regime is**

$$\Gamma = -\frac{\pi}{4}\sqrt{2\varepsilon}\,\frac{2}{|\varepsilon'|}\frac{cM}{|e|\chi'}\frac{N\mathrm{v}_t^2}{\pi^{3/2}}\left[\eta_1\left(\frac{p'}{p}+\frac{e\Phi'}{T}\right)+\eta_2\frac{T'}{T}\right],$$

where η_1 and η_2 are coefficients that are nonlinear function of the radial electric field.

- **The nonlinearity onsets when $\left|\frac{cE_r}{B_p\mathrm{v}_\mathrm{t}}\frac{2/\varepsilon}{\rho_p(d\ln\varepsilon/dr)}\right| > 1$.**

Conclusions

- **There are two symmetry-breaking mechanisms that are important when there are MHD activities or error fields present. One is the direct addition and the other is the distortion of the magnetic surface.**

- **There are two classes of the neoclassical toroidal plasma viscosity depending on whether the momentum restoring term in the collision operator is important or not.**

- **In both cases, neoclassical toroidal plasma viscosity has complex dependences on plasma parameters and can be tested under a variety of experimental conditions.**

- **Most importantly, neoclassical plasma viscosity is a nonlinear function of the radial electric field and its magnitude decreases when the radial electric field increases.**

- **The transport coefficients in the superbanana regime do not depend explicitly on the equilibrium magnetic field and can be quite large in magnitude even with relatively small perturbations.**