



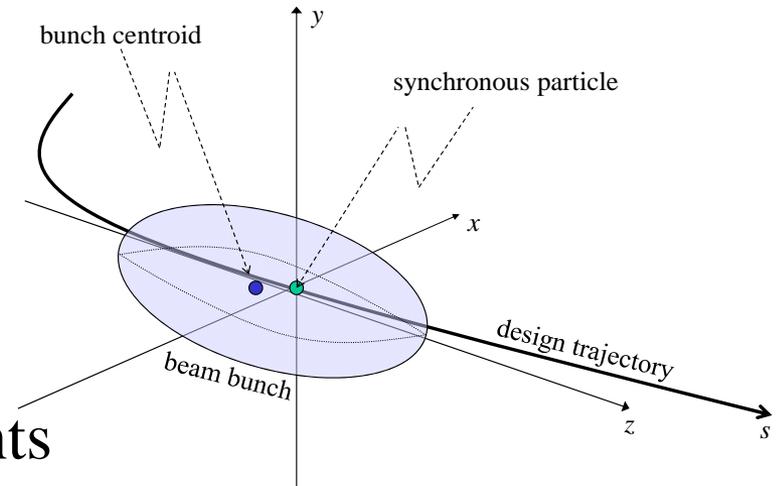
Control Room Accelerator Physics

Day 2

Linear Beam Optics

Outline

1. Motivation
2. The beam frame
3. Computing beamline elements
4. Modeling the beamline
5. Summary



Linear Beam Optics

Overview

- Accelerator systems employ a full complement of magnetic and RF devices for their operation
 - The RF devices are used for acceleration
 - The magnetic devices are used for transport and containment
- Here we focus on magnets; they define the *beamline*
 - Motion of charged particles through these magnets is analogous to the behavior of light rays through *optical* devices
- Consider only the 1st order forces of magnetic fields
 - The resulting equations of motion are *linear* – Matrices!
- Some magnets are specifically 2nd order and higher
 - We shall not consider these in *linear* beam optics

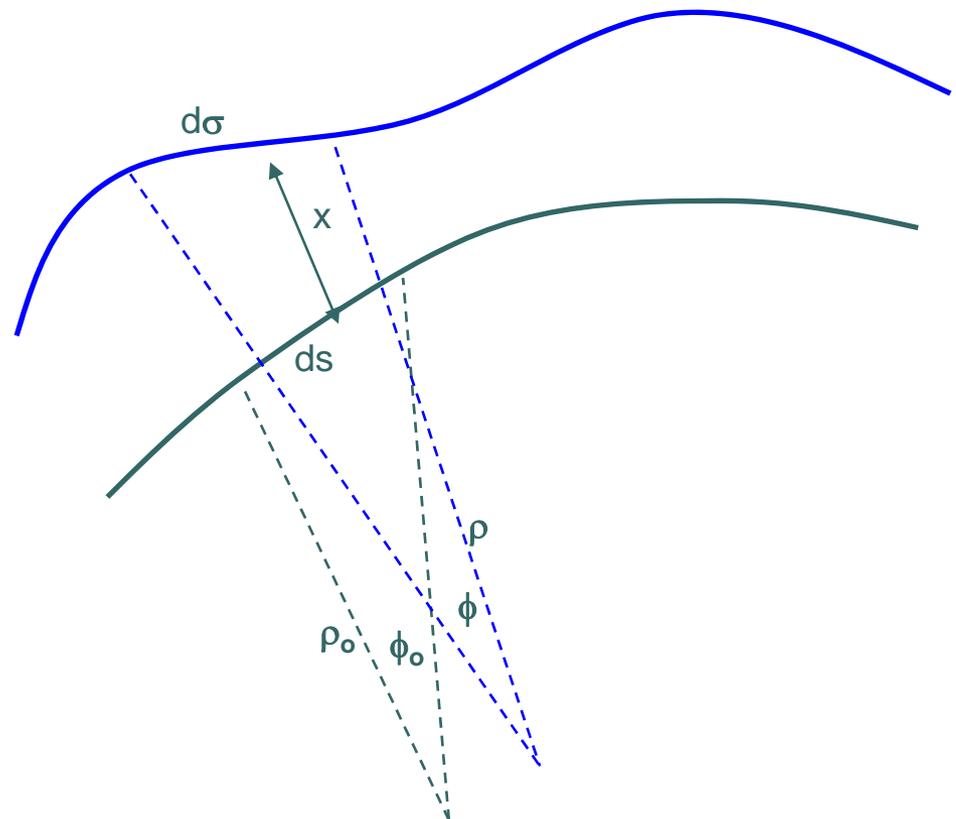
Motivation

When designing an accelerator, the first step is to define a *reference trajectory*. Only the *ideal, or synchronous, particle* actually follows this trajectory. All other beam particles follow trajectories about the design trajectory.

What we really need is a model describing particle deviations from the reference trajectory.

The idea is to subtract the reference trajectory from the actual trajectory, and then take the linear approximation by discarding all higher order terms...

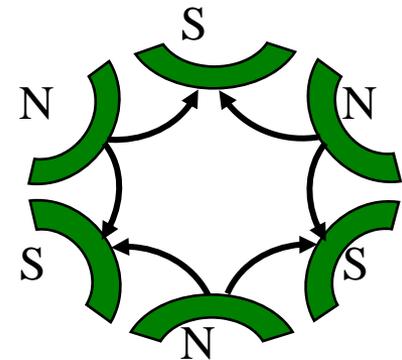
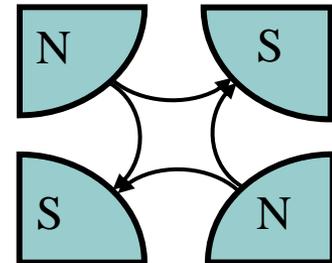
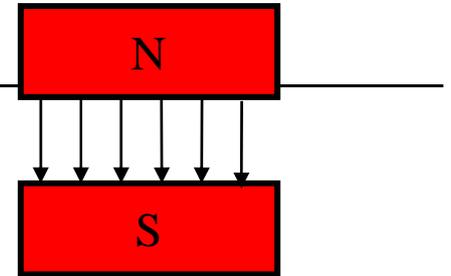
This gives us equations for “ x , y , and z ”, which are the particle coordinates about the reference trajectory.



Implementing the Beamline

Beamline Elements

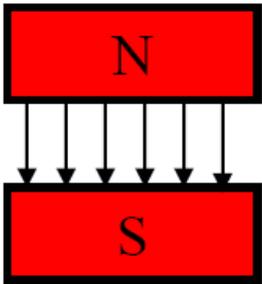
- Once the reference trajectory is agreed upon we employ a series of magnets to define the physical beamline
 - These magnets are typically “multi-pole” structures which produce the simplest type of fields possible
 - Simplifies analysis
 - Reduces beam degradation
 - The fields, in turn, produce the required forces on the beam particles



Magnet Families

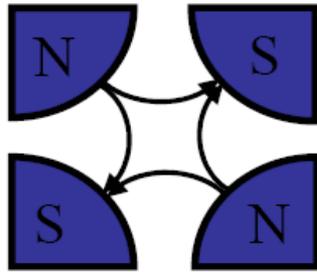
We name magnet types by the number n of pole faces from which it is composed.

$n=1$: Dipole



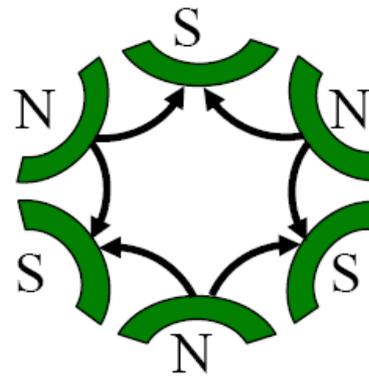
180° between poles

$n=2$: Quadrupole



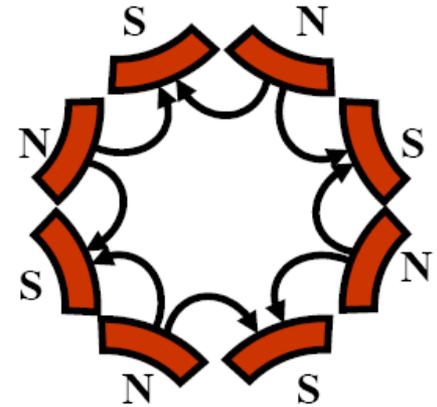
90° between poles

$n=3$: Sextupole



60° between poles

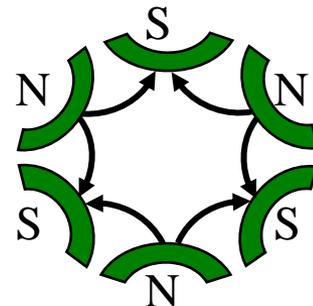
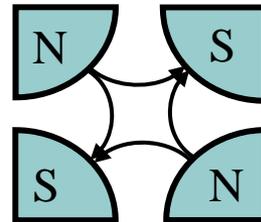
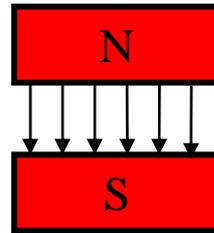
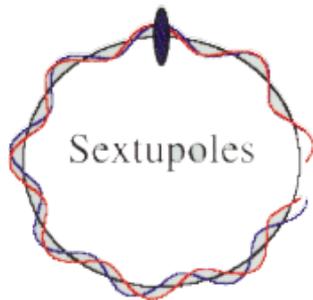
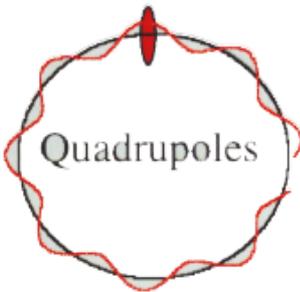
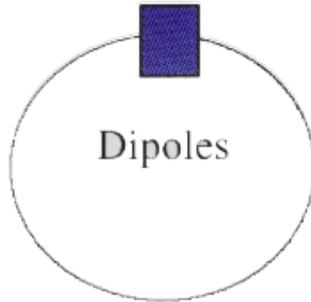
$n=4$: Octupole



45° between poles

- In general, poles are equally spaced and $360^\circ/2n$ apart.
- The pole faces and their spacing determine the magnetic field pattern
- The “skewed” version of the magnet is obtained by rotating magnet by $180^\circ/2n$.
- The field patterns determine the forces applied on the beam particles.

Multi-pole Magnet Applications within the Beamline



Bending (following reference trajectory)

Focusing the beam

"Chromatic compensation"

● ● ● | Magnet Photos



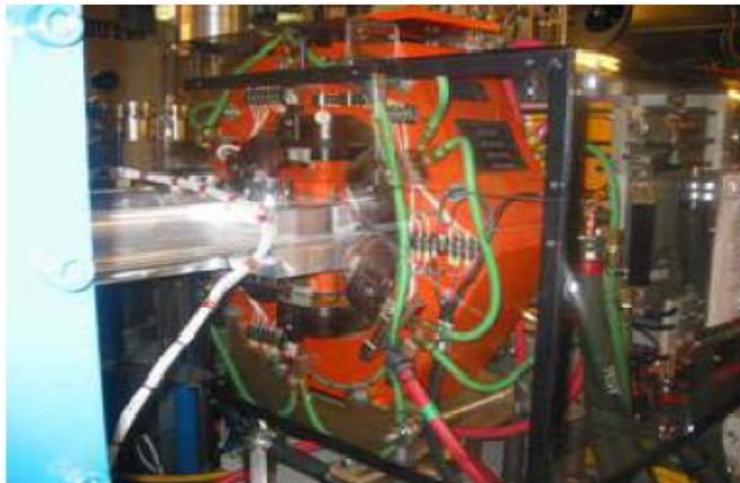
SNS ring dipole

Dipole



Quadrupole

Sextupole



Magnet Forces

From Physics to Engineering: Designing the Particle Forces

- Magnets create magnetic field $\mathbf{B}(x,y,z)$
 - For field-free region of multi-pole, field equations are Laplace's Eq.

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

$$\mathbf{B} = \nabla \phi = \begin{pmatrix} \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{pmatrix}$$

- Fields create a force \mathbf{F} on particle with velocity \mathbf{V} at position (x,y,z) from the Lorentz force law

$$\mathbf{F}(x, y, z) = q\mathbf{V} \times \mathbf{B}(x, y, z), \quad \mathbf{V} = (\dot{x} \quad \dot{y} \quad \dot{z})$$

- Forces then affect the motion of the particles through Newton's equations of motion (or Hamiltonian dynamics)

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}(x, y, z), \quad \mathbf{P} = \gamma m \mathbf{V}$$

Magnetic Forces

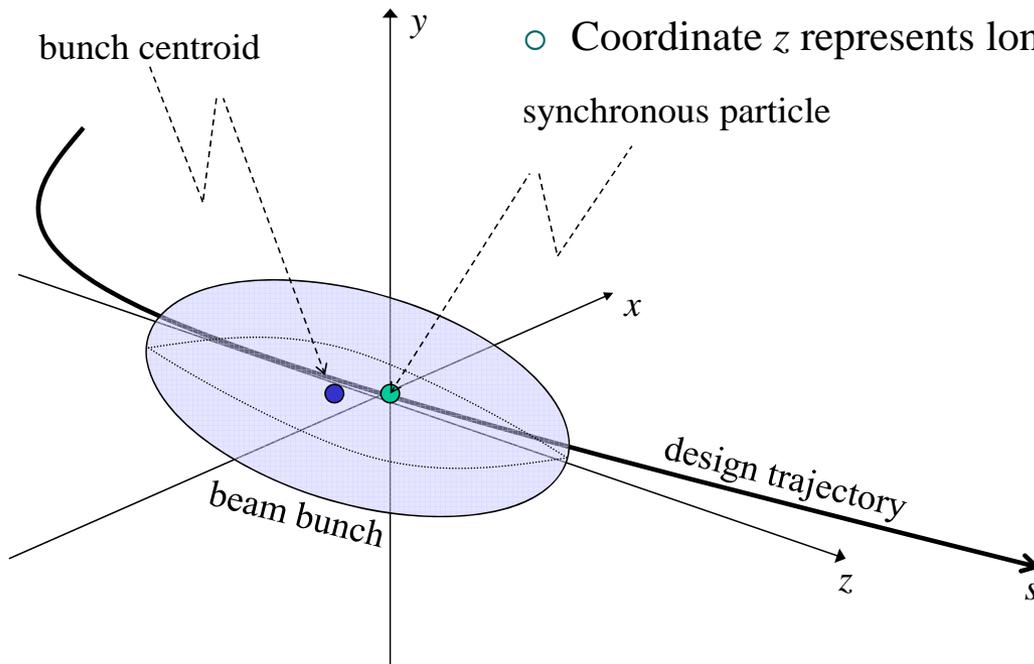
Designing the Beamline

- It is up to the skill of the designer to
 1. Create a magnet with high-quality fields using the field equations
 2. Align the magnet to produce the correct force on the particle using the Lorentz force law
 3. Ensure that the particles are deflected appropriately using the equations of motion

Modeling the Reference Trajectory

Beam Frame Coordinates

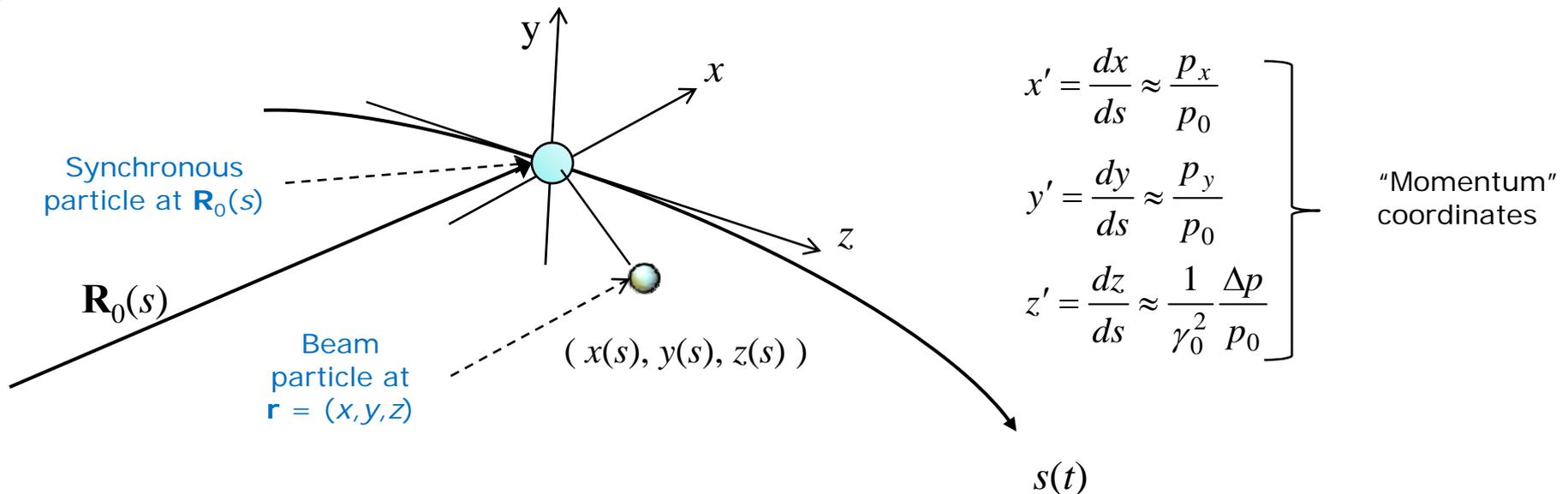
- Now consider a special coordinate system (x,y,z) that follows the *synchronous particle* along the *design orbit* parameterized by path length s . We call it the *beam frame*.
 - Synchronous particle located at origin $(x,y,z) = (0,0,0)$
 - Coordinates x and y are horizontal and vertical offsets, resp.
 - Coordinate z represents longitudinal offset from origin



Modeling the Reference Trajectory

The “Beam Frame”

- The beam frame is **not** an inertial frame
 - laboratory frame that follows synchronous particle
 - All other beam particles are identified by their coordinates in this frame
 - We can Lorentz transform to the stationary beam frame if necessary
 - (This is done for space charge calculations)



Longitudinal Coordinate

Longitudinal Coordinate Conventions

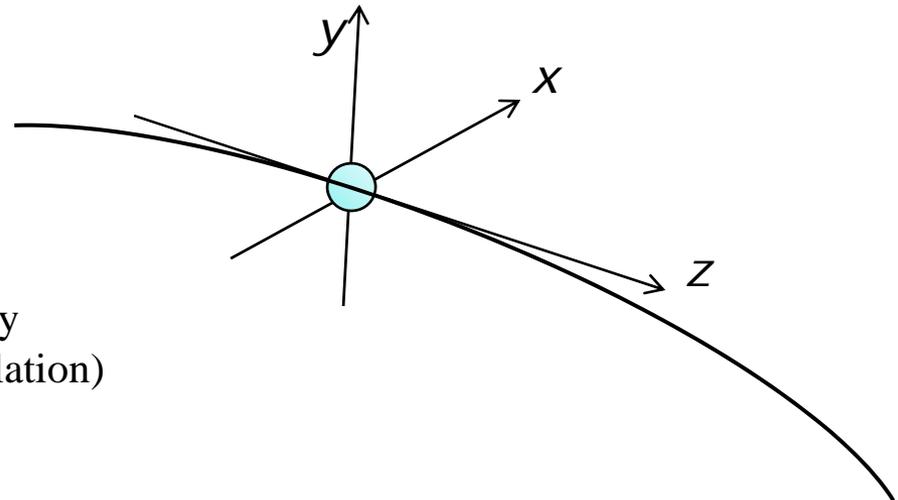
- Phase space coordinate z' is different than standard momentum offset parameter δ
 - Coordinate z' represents positional drift away from origin with s
 - Coordinate δ represents the fraction off mechanical momentum
 - They are related according to

$$\begin{aligned}\delta &\equiv \frac{\Delta p}{p_0} = \frac{p - p_0}{p_0} \\ &= \gamma_0^2 z'\end{aligned}$$

- Factor γ_0^2 from special relativity (length contraction and time dilation)

- Note finally

$$\dot{z} = \dot{s} + \dot{s}z' = \beta_0 c(1 + z')$$



Recall Longitudinal Coordinate

ONCE AGAIN:

Longitudinal coordinate is special because it points in direction of propagation

Special Relativity must be considered

- Alternate representations longitudinal “momentum”
 - Longitudinal divergence angle $z' \equiv dz/ds$
 - Off-momentum parameter $\delta \equiv (p - p_0)/p_0$
 - Energy difference parameter $\Delta W \equiv W - W_0$

XAL uses z'

Longitudinal Momentum

Comparison

Longitudinal Divergence z'

- $z' \rightarrow 0$ as $\gamma_0 \rightarrow \infty$ (as $v \rightarrow c$)
- Symmetric dynamics equations
- Convenient for space charge
- Numerically sensitive at high energy
- Probably better suited for protons

Momentum difference δ

- δ can remain finite even if $\gamma_0 \rightarrow \infty$
- Asymmetric dynamics equations
- Convenient for dispersion
- Numerically stable at high energy
- Probably better suited for electrons

Modeling the Beamline

Transfer Matrix Approach to Linear Beam Optics

- Because we only consider linear forces we can use linear algebra
- We represent each beamline element as a (transfer) matrix with parameters identifying
 - Magnet strength
 - Magnet length
 - Etc.
- Beam Particle Coordinates
 - Each particle will be represented as a point in the six-dimension phase space
 - $\mathbf{z} = (x, x', y, y', z, z')$
 - Follows from the three spatial coordinates and the three corresponding momentum coordinates required for a solution to eq.s of motion

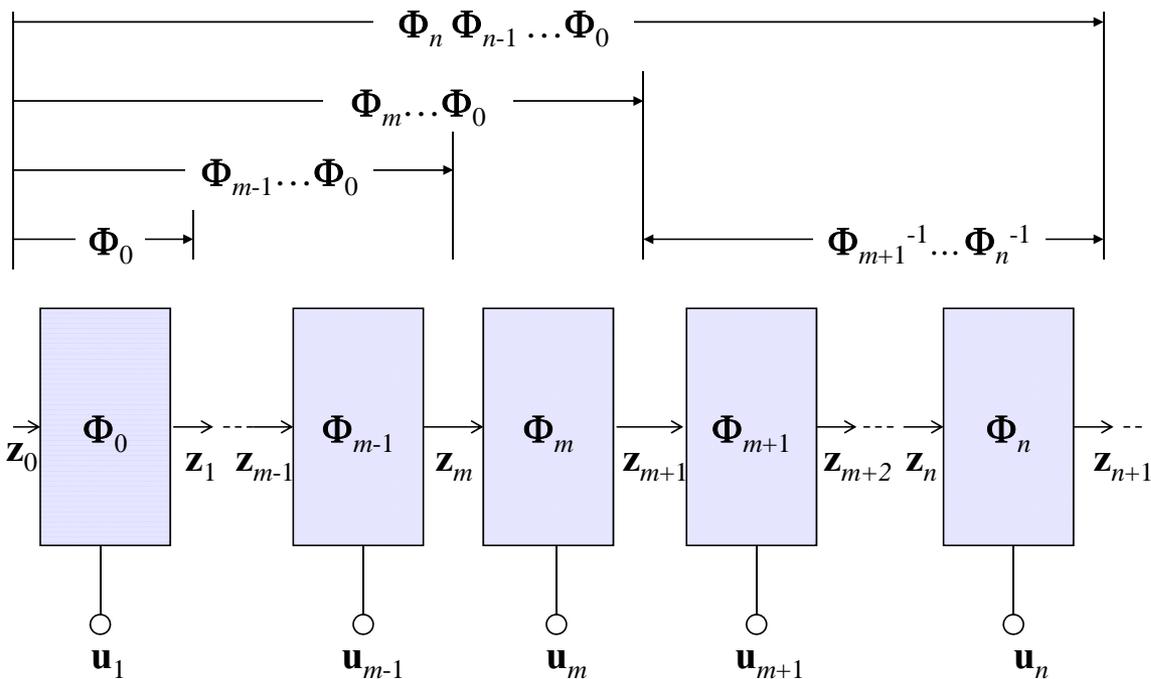
Our task then, is to start from the equations of motion and derive a matrix for each element which includes the effects of motion in a curva-linear coordinate system

(State space representation)

Motivation

Modeling the Entire Beamline

- One we have each beamline element n represented (mathematically) as a transfer matrix Φ_n , the entire beamline can be modeled as a cascade of these elements; multiplication of the $\{\Phi_n\}$ being the dynamics
 - We can propagate forward or backward through the network using the $\{\Phi_n\}$



Note that each beamline element may also have a control parameter \mathbf{u} associated with it.

Modeling the Beamline

Strategy for Transfer Matrix Model Derivation

- The computational details for determining each beamline element matrix can be found in many excellent texts in accelerator physics.
 - We will not present these derivations here
- We focus on a (somewhat nonstandard) high-level picture of how these computation follow
 - In some global coordinate system (where the beamline is defined) we define the design trajectory as the spatial curve $\mathbf{R}_0(s)$
 - We form a moving coordinate frame (x,y,z) with origin at $\mathbf{R}_0(s)$ about which beam particles have coordinates $\mathbf{r} = (x,y,z)$ within that frame
 - We write the equations of motion in the global frame, then take only first-order terms of the forces
 - We then translate the forces into the new moving frame
 - From there, the matrices for individual elements can be derived without regard to the original (global) beamline

Modeling the Beamline Elements

Equations of Motion (cont.)

After (extensive) manipulation, the equations of motion can be put into the form

$$\mathbf{z}' = \mathbf{G}(s)\mathbf{z} - \mathbf{\Gamma}(s)\mathbf{z}$$

where

$$\mathbf{z} \equiv \begin{pmatrix} \mathbf{r} \\ \mathbf{v} \end{pmatrix}, \quad \mathbf{\Gamma}(s) = \begin{pmatrix} \mathbf{\Omega}(s) & \mathbf{0} \\ \mathbf{0} & \mathbf{\Omega}(s) \end{pmatrix}, \quad \mathbf{G}(s) \equiv \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \frac{1}{\beta c p_0(s)} \mathbf{G}_r(s) & \frac{1}{\beta c p_0(s)} \mathbf{G}_v(s) \end{pmatrix}$$

\mathbf{z} is the phase coordinate vector in the moving frame (re-ordered coordinates)

$\mathbf{\Gamma}$ is the connection (centripetal, Coriolis, etc.) for the rotating frame

\mathbf{G} is the 1st order force matrix in the moving frame

In linear accelerators (linacs) $\mathbf{\Gamma} = \mathbf{0}$ since the frame (x,y,z) does not rotate

Note here we have separated the position coordinate $\mathbf{r} = (x,y,z)$ from the momentum coordinates $\mathbf{v} = (x',y',z')$

Modeling the Beamline Elements

Transfer Matrices

- Typically the matrices $\Gamma(s)$ and $\mathbf{G}(s)$ are constant within a beamline element (e.g., magnet)

- Thus, the solution to

$$\mathbf{z}' = (\mathbf{G} - \Gamma)\mathbf{z}$$

is given by the matrix exponential $\mathbf{z}(s) = e^{s(\mathbf{G}-\Gamma)} \mathbf{z}_0$ where $\mathbf{z}_0 = \mathbf{z}(0)$

- For particle motion within a beamline element n we have

$$\mathbf{z}(s) = \Phi_n(s)\mathbf{z}_0 \quad \Phi_n(s) \equiv e^{s(\mathbf{G}_n - \Gamma_n)}$$

where

\mathbf{G}_n is the *generator matrix* for element n (the applied forces)

Γ_n is the *coordinate connection* for element n (inertial forces)

Φ_n is the *transfer matrix* for element n

Modeling the Beamline and Elements

Summary

- The beamline is designed according to a *reference orbit* $\mathbf{R}_0(s)$ and *synchronous velocity* $\mathbf{V}_0(s)$ for distance s along the beamline.
 - The quantities $\mathbf{R}_0(s)$ and $\mathbf{V}_0(s)$ are the position and velocity of the *synchronous particle* in the *global coordinates* (X, Y, Z) at a distance s .
- All other particles will propagate down the beamline in the vicinity of the synchronous particle. To model this motion, we...
 - Construct a moving, *local coordinate system* (x, y, z) about the synchronous particle to describing the relative position of the beam particles.
 - Translate the 1st order forces of the magnets to this frame and also account for the internal forces of the moving frame with the connection Γ .
 - Express the equation of motion in the local frame. The result is...
A matrix-vector equation for the generalized particle phase coordinates \mathbf{r} and \mathbf{v} .

Modeling the Beamline

Practical Considerations

- In general we let the theoreticians and engineers compute the transfer matrices and connections for the beamline.
 - Then we use the results for our model reference control and other engineering applications
- However, there are some simple cases which we can compute that are instructive for observing the process.
 - Drift space – a field free region in a straight section of beamline
 - Quadrupole magnet – a magnet in a straight section of beamline

Beamline Element Transfer Matrices

Example: Drift Space

- The beamline is straight; therefore $\Gamma(s) = \mathbf{0}$
- No fields are present, thus, \mathbf{G}_r and \mathbf{G}_v are both zero so the matrix \mathbf{G} is

$$\mathbf{G} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{G}_r & \mathbf{G}_v \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

- The solution for $\Phi_{drift}(s)$ is

$$\Phi_{drift}(s) = e^{s\mathbf{G}} = \mathbf{I} + s\mathbf{G} + \frac{s^2}{2}\mathbf{G}^2 + \frac{s^3}{2 \cdot 3}\mathbf{G}^3 + \dots$$

However, the matrix \mathbf{G} is idempotent, that is, $\mathbf{G}^2 = \mathbf{0}$. Thus $e^{s\mathbf{G}}$ is easy to compute.

- Thus, the solution is $\Phi_{drift}(s) = \mathbf{I} + s\mathbf{G}$, or explicitly

$$\Phi_{drift}(s) = \mathbf{I} + s\mathbf{G} \equiv \begin{pmatrix} 1 & 0 & 0 & s & 0 & 0 \\ 0 & 1 & 0 & 0 & s & 0 \\ 0 & 0 & 1 & 0 & 0 & s \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

This form differs slightly from others in the literature because we have taken $\mathbf{z} = (x, y, z, x', y', z')$ rather than $\mathbf{z} = (x, x', y, y', z, z')$

Beamline Element Transfer Matrices

Example: Drift Space (cont.)

The single particle phase trajectory through a field free region (drift space) is then given by the matrix-vector equation

$$\mathbf{z}(s) = \mathbf{\Phi}_{drift}(s)\mathbf{z}_0$$

where \mathbf{z}_0 is the phase coordinate at the entrance of the drift and

$$\mathbf{\Phi}_{drift}(s) = \begin{pmatrix} 1 & 0 & 0 & s & 0 & 0 \\ 0 & 1 & 0 & 0 & s & 0 \\ 0 & 0 & 1 & 0 & 0 & s \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} x \\ y \\ z \\ x' \\ y' \\ z' \end{pmatrix}$$

is the *transfer matrix* for the drift space.

Beamline Element Transfer Matrices

Example: Quadrupole Lens

- Ideal quadrupole magnets have fields defined by

$$\mathbf{B}(\mathbf{R}) = (B_x \quad B_y \quad B_z) = (G(Y - Y_0) \quad G(X - X_0) \quad 0)$$

(in global coordinates) where G is the *field gradient* of the quadrupole

- The beamline is straight, thus $\mathbf{O}(s) = \mathbf{I}$ and $\Gamma(s) = 0$
- Computing the force differentials yields

$$\mathbf{G}_r = \frac{\partial F(\mathbf{R}_0, \mathbf{V}_0)}{\partial \mathbf{R}} = \frac{\partial}{\partial \mathbf{R}} q(V_x \quad V_y \quad V_z) \times (GY \quad GX \quad 0) \Big|_{\mathbf{V}=\mathbf{V}_0} = q\beta c G \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{G}_v = \frac{\partial F(\mathbf{R}_0, \mathbf{V}_0)}{\partial \mathbf{V}} = \frac{\partial}{\partial \mathbf{V}} q(V_x \quad V_y \quad V_z) \times (GY \quad GX \quad 0) \Big|_{\mathbf{R}=\mathbf{R}_0} = \mathbf{0}$$

Beamline Element Transfer Matrices

Example: Quadrupole Lens (cont.)

- The external force generator matrix is then

$$\mathbf{G}(s) = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{G}_r & \mathbf{G}_v \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -k^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & k^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- where $k^2 = qG/p_0$ is the focusing constant
- The transfer matrix is given by $\Phi_{quad}(s) = e^{s\mathbf{G}}$

Beamline Element Transfer Matrices

Example: Quadrupole Lens (cont.)

- The result is

$$\Phi_{quad}(s) = e^{s\mathbf{G}} = \begin{pmatrix} \cos ks & 0 & 0 & \frac{\sin ks}{k} & 0 & 0 \\ 0 & \cosh ks & 0 & 0 & \frac{\sinh ks}{k} & 0 \\ 0 & 0 & 1 & 0 & 0 & s \\ -k \sin ks & 0 & 0 & \cos ks & 0 & 0 \\ 0 & k \sinh ks & 0 & 0 & \cosh ks & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Once again this form differs slightly from others in the literature because we have taken $\mathbf{z} = (x, y, z, x', y', z')$ rather than $\mathbf{z} = (x, x', y, y', z, z')$

Beamline Element Transfer Matrices

Example: Quadrupole Lens (cont.)

- In the conventional phase space coordinate ordering the quadrupole transfer matrix appears as

$$\Phi_Q(s) \equiv \begin{pmatrix} \cos ks & \frac{\sin ks}{k} & 0 & 0 & 0 & 0 \\ -k \sin ks & \cos ks & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh ks & \frac{\sinh ks}{k} & 0 & 0 \\ 0 & 0 & k \sinh ks & \cosh ks & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & s \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $k \equiv (qG/p_0)^{1/2}$

- It is also possible to compute this result using standard techniques from ODE

Modeling the Entire Beamline

Transfer Matrix Equations

- The transfer matrix $\Phi_n(s)$ for element n describes the element's effects upon the state vector \mathbf{z} of the beam particle through the length of the element.
 - If the element has length l_n then $\Phi_n(l_n)$ is the effect of passing completely through the element.
 - We can cascade each beamline element passing the beam state from the exit of one to the entrance of the next to simulate the effects of the entire beamline.
 - The resulting modeling equations are

$$\mathbf{z}_{n+1} = \Phi_n \mathbf{z}_n$$

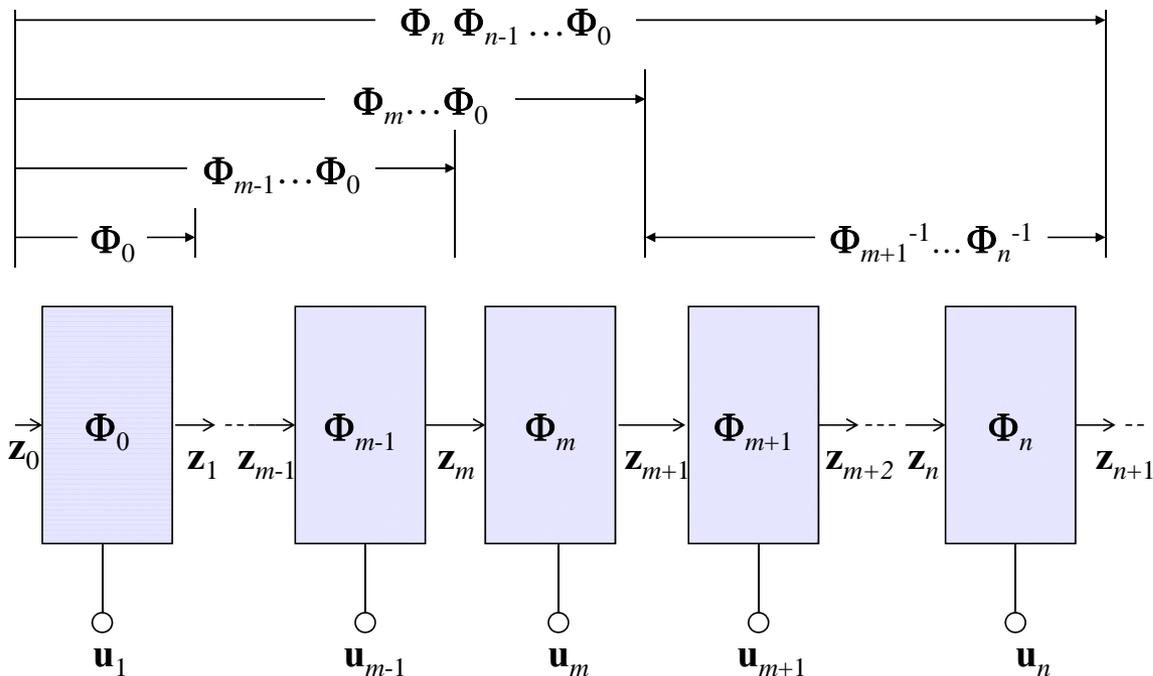
where \mathbf{z}_n is the beam state at the entrance to element n and Φ_n is shorthand for $\Phi_n(l_n)$.

- This formula describes the internal workings of the XAL online model

Modeling the Entire Beamline

Multi-stage control network

- In controls parlance we call such a configuration a *multi-stage control network*
 - We can propagate forward or backward through the network using the $\{\Phi_n\}$

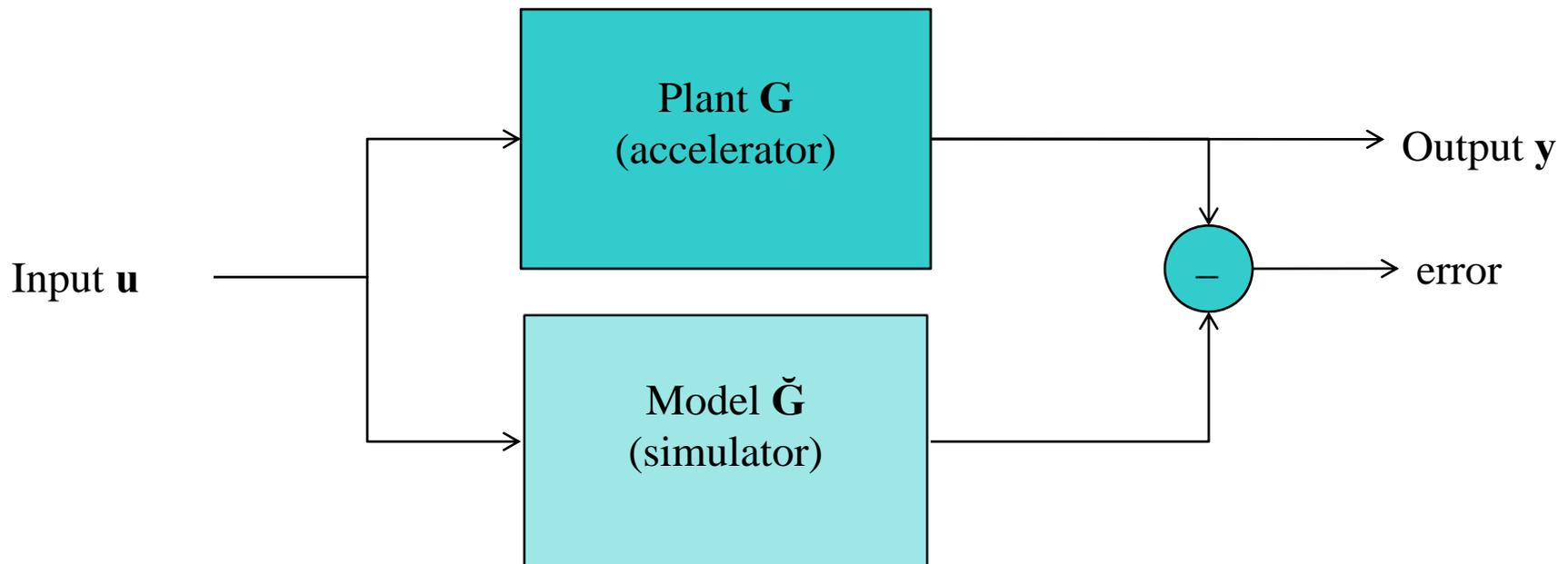


Note that each beamline element may also have a control parameter \mathbf{u} associated with it.

Modeling the Entire Beamline

Applications: Model Reference Control

- If we have model of the beamline, that we can compare actual measurements from the machine to simulated ones from the online model. We can use the differences in these values to make inference about the operation of the machine.



Beamline Modeling

Beam Steering from a Control Theoretic Perspective

- Say we have Beam Position Monitors (BPMs) as our sensors, then our observables are the coordinates (x,y,z) ; that is, we do not have access to the full state vector – no momentum components

- Set $\mathbf{y} \equiv (x \ y \ z)^T$

- Then $\mathbf{y}_n = \mathbf{C}\mathbf{z}_n(t)$

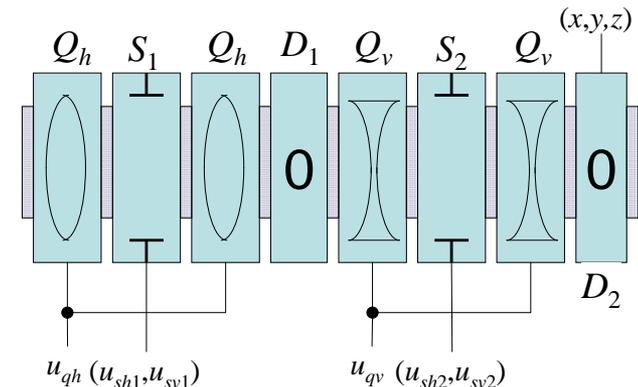
where

$$\mathbf{C} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

- Our modeling equations are then

$$\mathbf{z}_{n+1} = \mathbf{\Phi}_n \mathbf{z}_n$$

$$\mathbf{y}_n = \mathbf{C}\mathbf{z}_n$$



These are in the form of the discrete state space representation we have seen in linear systems

Summary

- We have a convenient method of modeling the beamline to 1st order
- This model provides a convenient reference for model reference control applications
- The model can be extended in a straightforward manner to handle the RMS statistics for a beam bunch.
- We can also treat the 1st order effects of space charge using this extension.

- Later we demonstrate how to instantiate an XAL online model that configures itself automatically for a selected AcceleratorSeq object.



Supplemental Material

- More details on Linear Beam Optics

Modeling the Beamline

Modeling the Particle Forces

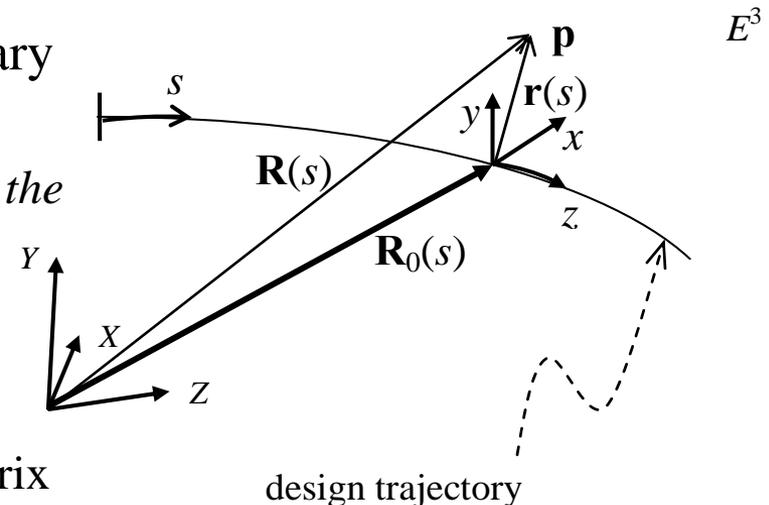
- Let the global coordinates be (X, Y, Z)
 - The design trajectory is represented by $\mathbf{R}_0(s) = (X(s), Y(s), Z(s))$

- Let $\mathbf{r} = (x, y, z)$ be the position of an arbitrary particle *in the beam frame*.
 - The position $\mathbf{R}(s)$ of the beam particle *in the global coordinates* is then

$$\mathbf{R}(s) = \mathbf{R}_0(s) + \mathbf{O}(s)\mathbf{r}$$

where $\mathbf{O}(s)$ in $SO(3)$ is an orthogonal matrix representing the orientation of (x, y, z) with (X, Y, Z)

- (Recall that $SO(3)$ is the set of 3D rotations)



Beamline Modeling

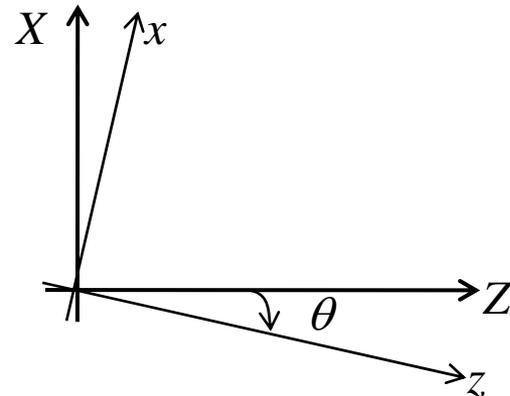
Example: Local to Global Coordinate Transform

- For example, for a bend in the horizontal direction the matrix $\mathbf{O}(s)$ in $SO(3)$ in $\mathbf{R}^{3 \times 3}$ appears as

$$\mathbf{O}(s) = \mathbf{O}_y(\theta(s)) \equiv \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}$$

where $\theta(s)$ is the angle between the x and X axes (or z and Z axes).

- Here $\mathbf{O}(s)$ is simply a clockwise rotation of (x,y,z) about the y axis.



Beamline Modeling

Velocity vector in beam coordinates: Covariant Formulation

To find the velocity component $\mathbf{V}(s)$ first note that

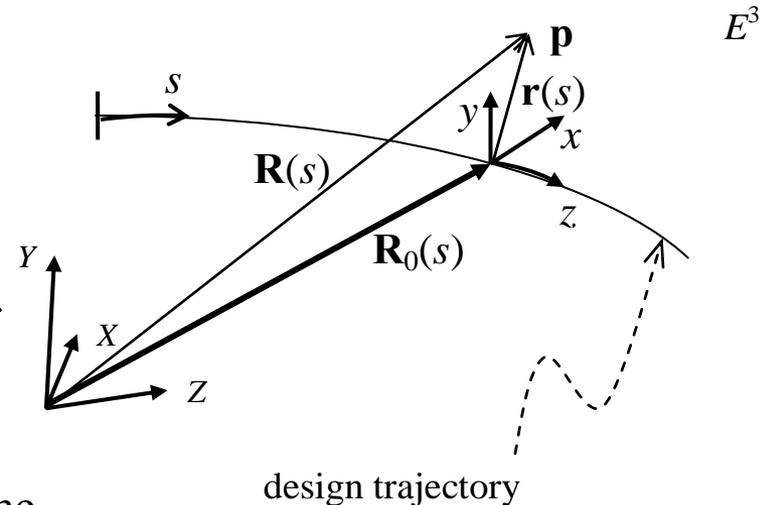
$$\mathbf{V}(s) = d\mathbf{R}/dt = d\mathbf{R}(s)/ds \cdot ds/dt = \mathbf{R}'(s) \cdot \beta c$$

To find the divergence $\mathbf{R}'(s)$ we differentiate $\mathbf{R}(s)$ with respect to s

$$\begin{aligned}\mathbf{R}'(s) &= \mathbf{R}'_0(s) + \mathbf{O}'(s)\mathbf{r} + \mathbf{O}(s)\mathbf{r}' \\ &= \mathbf{R}'_0(s) + \mathbf{O}(s)\left(\mathbf{O}^T(s)\mathbf{O}'(s)\mathbf{r} + \mathbf{r}'\right)\end{aligned}$$

- Let $\mathbf{V}_0(s) \equiv \beta c \mathbf{R}'_0(s)$ be the velocity of the synchronous particle
- Let $\mathbf{v} \equiv \beta c \mathbf{O}^T(s)\mathbf{O}'(s)\mathbf{r} + \beta c \mathbf{r}'$ be the particle velocity w.r.t. the moving frame
- Define $\mathbf{\Omega}(s) \equiv \mathbf{O}^T(s)\mathbf{O}'(s)$ then

$$\mathbf{v} = \beta c \mathbf{\Omega}(s)\mathbf{r} + \beta c \mathbf{r}'$$



Modeling Beam Dynamics

The Covariant Formulation

- The matrix $\mathbf{\Omega}(s)$ is known as a connection and contains the inertial effects of the moving frame (e.g., centripetal forces, Coriolis forces)
 - Since we know $\mathbf{O}(s)$ from the design trajectory $\mathbf{R}_0(s)$, we can always compute $\mathbf{\Omega}(s) \equiv \mathbf{O}^T(s)\mathbf{O}'(s)$ *a priori*.
 - This is the called the *covariant formulation*.
 - *Within the beam frame* we can replace the differentiation operator d/ds with $\mathbf{\Omega}(s) + d/ds$ to obtain the *frame-invariant* equations of motion.
 - For example, we can find the acceleration vector $\mathbf{A}(s)$ in the global coordinates using the same procedure...

Modeling the Beamline

Acceleration Vector: Covariant Formulation

- Starting from the velocity vector $\mathbf{V}(s)$ given by

$$\mathbf{V}(s) \equiv \beta c \mathbf{R}'(s) = \mathbf{V}_0(s) + \mathbf{O}(s) \mathbf{v}$$

we have

$$\mathbf{A}(s) \equiv \beta c \mathbf{V}'(s) = \beta c \mathbf{V}'_0(s) + \mathbf{O}(s) (\beta c \boldsymbol{\Omega}(s) \mathbf{v} + \beta c \mathbf{v}')$$

- Collecting everything

Global

$$\mathbf{R} = (X, Y, Z)$$

$$\mathbf{R}(s) = \mathbf{R}_0(s) + \mathbf{O}(s) \mathbf{r},$$

$$\mathbf{V}(s) = \mathbf{V}_0(s) + \mathbf{O}(s) \mathbf{v}, \quad \mathbf{V}_0 \equiv \beta c \mathbf{R}'_0$$

$$\mathbf{A}(s) = \mathbf{A}_0(s) + \mathbf{O}(s) \mathbf{a}, \quad \mathbf{A}_0 \equiv \beta c \mathbf{V}'_0$$

Local

$$\mathbf{r} = (x, y, z)^T$$

$$\mathbf{v} = \beta c \boldsymbol{\Omega}(s) \mathbf{r} + \beta c \mathbf{r}'$$

$$\mathbf{a} = \beta c \boldsymbol{\Omega}(s) \mathbf{v} + \beta c \mathbf{v}'$$

Connection

$$\boldsymbol{\Omega}(s) = \mathbf{O}^T(s) \mathbf{O}'(s)$$

Beamline Modeling

Example: Horizontal Bend

$$\mathbf{R}_0(s) = \rho(\cos \theta(s) \quad 0 \quad \sin \theta(s)) \quad \text{where} \quad \theta(s) \equiv sh = s / \rho$$

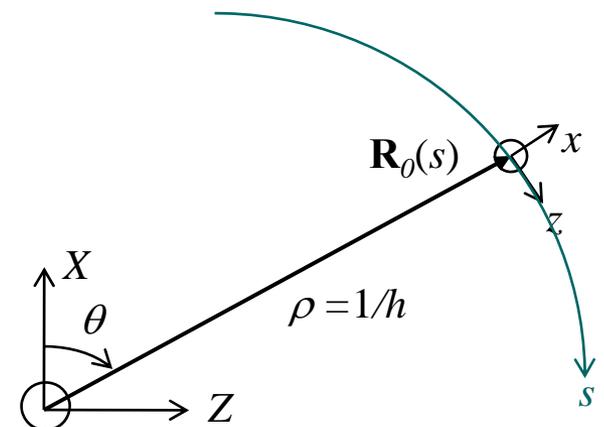
$$\mathbf{O}(s) = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} = e^{sh\mathbf{L}_y}, \quad \text{where} \quad \mathbf{L}_y \equiv \mathbf{O}'(0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{\Omega}(s) = \mathbf{O}^T(s)\mathbf{O}'(s) = \left[e^{-sh\mathbf{L}_y} \left[h\mathbf{L}_y e^{sh\mathbf{L}_y} \right] \right] = h\mathbf{L}_y e^{-sh\mathbf{L}_y} e^{sh\mathbf{L}_y} = h\mathbf{L}_y$$

$$\mathbf{r} = (x, y, z)$$

$$\mathbf{v} = \beta ch\mathbf{L}_y \mathbf{r} + \beta c\mathbf{r}'$$

$$\mathbf{a} = \beta ch\mathbf{L}_y \mathbf{v} + \beta c\mathbf{v}'$$



Modeling the Beamline Elements

First-Order Forces

- The global particle force \mathbf{F} is a function of particle position \mathbf{R} and particle velocity \mathbf{V} ; that is

$$\mathbf{F} = \mathbf{F}(\mathbf{R}, \mathbf{V}) = \mathbf{F}(\mathbf{R}_0 + \mathbf{O}\mathbf{r}, \mathbf{V}_0 + \mathbf{O}\mathbf{v})$$

Expanding

$$\mathbf{F}(\mathbf{R}, \mathbf{V}) = \mathbf{F}(\mathbf{R}_0, \mathbf{V}_0) + \frac{\partial \mathbf{F}(\mathbf{R}_0)}{\partial \mathbf{R}} \mathbf{O}\mathbf{r} + \frac{\partial \mathbf{F}(\mathbf{V}_0)}{\partial \mathbf{V}} \mathbf{O}\mathbf{v} + \mathcal{O}(|\mathbf{r}|^2) + \mathcal{O}(|\mathbf{v}|^2)$$

Modeling the Beamline Elements

Equations of Motion

- The mechanical momentum $\mathbf{P} = (P_x, P_y, P_z)$ for a beam particle (in global coordinates) is

$$\mathbf{P} \equiv \gamma m \mathbf{V}$$

where

γ is the relativistic factor

m is particle mass

$\mathbf{V} = (V_x, V_y, V_z)$ is particle velocity

If there is no acceleration, then β' and γ' are zero

- The equations of motion for a beam particle with momentum \mathbf{P} are

$$\mathbf{F} = \frac{d\mathbf{P}}{dt} = \gamma m \mathbf{V}' \beta c = \beta \gamma m c \mathbf{A} = \beta \gamma m c (\mathbf{A}_0 + \mathbf{Oa})$$

Modeling the Beamline Elements

Equations of Motion (cont.)

- Combining the previous equation with the force expansion out to first order produces

$$\beta\gamma mc(\mathbf{A}_0 + \mathbf{O}\mathbf{a}) \approx \mathbf{F}(\mathbf{R}_0, \mathbf{V}_0) + \frac{\partial \mathbf{F}(\mathbf{R}_0, \mathbf{V}_0)}{\partial \mathbf{R}} \mathbf{O}\mathbf{r} + \frac{\partial \mathbf{F}(\mathbf{R}_0, \mathbf{V}_0)}{\partial \mathbf{V}} \mathbf{O}\mathbf{v}$$

- Note that $\beta\gamma mc\mathbf{A}_0(s) = \mathbf{F}[\mathbf{R}_0(s), \mathbf{V}_0(s)]$ is the equation of the synchronous particle, that is, this equation defines the design trajectory $\mathbf{R}_0(s)$ and the design velocity $\mathbf{V}_0(s)$.
 - We may remove it from the above since it is already known
- The remaining component is the 1st order equations of motion

$$\mathbf{a} \approx \frac{1}{\beta\gamma mc} \mathbf{O}^T \frac{\partial \mathbf{F}(\mathbf{R}_0, \mathbf{V}_0)}{\partial \mathbf{R}} \mathbf{O}\mathbf{r} + \mathbf{O}^T \frac{\partial \mathbf{F}(\mathbf{R}_0, \mathbf{V}_0)}{\partial \mathbf{V}} \mathbf{O}\mathbf{v}$$

Modeling the Beamline Elements

Equations of Motion (cont.)

- To make the previous equation more transparent define the following:

$$\mathbf{G}_r(s) \equiv \mathbf{O}^T(s) \frac{\partial \mathbf{F}(\mathbf{R}_0(s), \mathbf{V}_0(s))}{\partial \mathbf{R}} \mathbf{O}(s)$$

$$\mathbf{G}_v(s) \equiv \mathbf{O}^T(s) \frac{\partial \mathbf{F}(\mathbf{R}_0(s), \mathbf{V}_0(s))}{\partial \mathbf{V}} \mathbf{O}(s)$$

- These matrices are the 1st order force differentials about the design trajectory in the beam frame coordinates (x, y, z) .
(Usually we have these explicitly.)
- The equations of motion are then

$$\mathbf{a} = \beta c \boldsymbol{\Omega}(s) \mathbf{v} + \mathbf{v}' \approx \frac{1}{\beta \gamma m c} \mathbf{G}_r(s) \mathbf{r} + \frac{1}{\beta \gamma m c} \mathbf{G}_v(s) \mathbf{v}$$

Modeling the Beamline Elements

Equations of Motion (cont.)

- The previous equation, along with the definition for \mathbf{v} provides a complete set for the description of beam-particle 1st order motion

$$\mathbf{r}' = -\boldsymbol{\Omega}(s)\mathbf{r} + \mathbf{I}\mathbf{v}$$

$$\mathbf{v}' = -\boldsymbol{\Omega}(s)\mathbf{v} + \frac{1}{\beta c p_0(s)} \mathbf{G}_r(s)\mathbf{r} + \frac{1}{\beta c p_0} \mathbf{G}_v(s)\mathbf{v}$$

- where $p_0(s) \equiv \beta(s)\gamma(s)mc$ is the design particle momentum.
- We can put this into matrix-vector form (recognize this form?)

$$\begin{pmatrix} \mathbf{r}' \\ \mathbf{v}' \end{pmatrix} = -\begin{pmatrix} \boldsymbol{\Omega}(s) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega}(s) \end{pmatrix} \begin{pmatrix} \mathbf{r} \\ \mathbf{v} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \frac{1}{\beta c p_0(s)} \mathbf{G}_r(s) & \frac{1}{\beta c p_0(s)} \mathbf{G}_v(s) \end{pmatrix} \begin{pmatrix} \mathbf{r} \\ \mathbf{v} \end{pmatrix}$$

Modeling the Beamline Elements

Equations of Motion (cont.)

- Even more compactly

$$\mathbf{z}' = \mathbf{G}(s)\mathbf{z} - \mathbf{\Gamma}(s)\mathbf{z}$$

where

$$\mathbf{z} \equiv \begin{pmatrix} \mathbf{r} \\ \mathbf{v} \end{pmatrix}, \quad \mathbf{\Gamma}(s) = \begin{pmatrix} \mathbf{\Omega}(s) & \mathbf{0} \\ \mathbf{0} & \mathbf{\Omega}(s) \end{pmatrix}, \quad \mathbf{G}(s) \equiv \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \frac{1}{\beta c p_0(s)} \mathbf{G}_r(s) & \frac{1}{\beta c p_0(s)} \mathbf{G}_v(s) \end{pmatrix}$$

\mathbf{z} is the phase coordinate vector in the moving frame (re-ordered coordinates)

$\mathbf{\Gamma}$ is the connection for the moving frame

\mathbf{G} is the 1st order force in the moving frame

In linear accelerators (linacs) $\mathbf{\Gamma} = \mathbf{0}$ since the frame (x,y,z) does not rotate

Modeling the Beamline Elements

Transfer Matrices

- Typically the matrices $\Gamma(s)$ and $\mathbf{G}(s)$ are constant within a beamline element (e.g., magnet)

- Thus, the solution to

$$\mathbf{z}' = (\mathbf{G} - \Gamma)\mathbf{z}$$

is given by the matrix exponential $\mathbf{z}(s) = e^{s(\mathbf{G}-\Gamma)} \mathbf{z}_0$ where $\mathbf{z}_0 = \mathbf{z}(0)$

- For particle motion within a beamline element n we have

$$\mathbf{z}(s) = \Phi_n(s)\mathbf{z}_0 \quad \Phi_n(s) \equiv e^{s(\mathbf{G}_n - \Gamma_n)}$$

where

\mathbf{G}_n is the *generator matrix* for element n (the applied forces)

Γ_n is the *coordinate connection* for element n (inertial forces)

Φ_n is the *transfer matrix* for element n